Resistance functions for two unequal spheres with Navier's slip boundary condition in linear flows at low Reynolds number

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Resistance functions for two spherical particles with Navier's slip boundary condition in general linear flows, including rigid translation, rigid rotation, and strain, at low Reynolds number are derived by the method of reflections as well as twin multipole expansions. In the solutions, particle radii and slip lengths can be chosen independently. In the course of calculations, single-sphere problem with slip boundary condition is solved by Lamb's general solution and the expression of multipole expansions, and Faxén's laws of force, torque, and stresslet for slip particle are also derived. The solutions of two-body problem are confirmed to recover the existing results for the no-slip limit and the case of equal scaled slip lengths.

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I. INTRODUCTION

According to increasing scientific interests in micro- and nanofluidics and nanotechnology in recent years, fluid mechanics is applied to such small-scale systems, in addition to molecular-level theories, where the characteristic Reynolds number is generally small enough to take the Stokes approximation governed by linear partial differential equations. In fluid mechanics, historically, both no-slip and slip boundary conditions were proposed in nineteenth century[1] when the proper boundary conditions were discussed in the first place. Navier[2] gave the slip boundary condition where the slip velocity is proportional to the tangential component of the surface force density. For gas flows, Maxwell[3] had shown that the surface slip is related to the non-continuous nature of the gas and the slip length is proportional to the mean-free path. For liquids, on the other hand, from experiments at that age, the no-slip boundary condition was accepted and since then had been treated as a fundamental law. However, by recent extensive studies on the surface slip in micro and nano scales, the physics of the liquid-solid slip is recognized to be much more complicated than that for gases. Actually apparent violations of the no-slip boundary condition at the liquid-solid interface in nano scale have been reported.[1,4–6]

Although the importance of the surface slip is realized, theoretical studies and analytical solutions for the slip boundary condition are very limited compared with those for the no-slip boundary condition. Basset solved the flow of single sphere with slip surface,[7] Felderhof derived Faxén's law and solutions expressed by multipole expansions for single sphere[8] and two spheres,[9] Bławzdziewicz *et al.* showed the interaction between the slip spheres and lubrication functions for the axisymmetric motion,[10] and Luo and Pozrikidis studied two slip spheres under the shear flow.[11] Recently, the present authors extended the Stokesian dynamics method (without lubrication) for slip particles using multipole expansions and Faxén's laws and obtained the slip dependencies for the drag coefficient and effective viscosity.[12] With no-slip boundary condition, the problem of two spherical particles is solved by Jeffrey and Onishi[13] and Jeffrey[14] for arbitrary size ratio of the particles in arbitrary linear flows. The extension to the slip particles was done by Ying and Peters[15] for the gas-solid system and by Keh and Chen[16] for the liquid-solid system, but they lack the strain flows. Keh and Chen[16] applied Navier's slip boundary condition under a condition that the ratios of the slip length and radius for two particles are equal.

In this paper, we will show the exact solution of two spheres in the form of resistance functions with arbitrary size ratio under Navier's slip boundary condition with arbitrary slip lengths in general linear flows including strain and shear flows. The present formulation is based on the no-slip case by Jeffrey and Onishi[13] and Jeffrey,[14] but we will show all the necessary equations in order that the present paper be self-contained. We refer equations in the references as Eq. (JO-1) for Jeffrey and Onishi,[13] Eq. (J-1) for Jeffrey,[14] and Eq. (KC-1) for Keh and Chen.[16]

The paper is organized as follows. In Sec II, the definition of resistance functions and Lamb's general solution are summarized. In Sec III, the solution of single sphere with slip boundary condition is shown. In Sec IV, two-body problem is solved first by method of reflections and then by twin multipole expansions. Concluding remarks are given in Sec V.

II. FORMULAE OF STOKES FLOWS

A. Resistance Functions

At low Reynolds number, the incompressible viscous fluid is governed by the Stokes equation

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \boldsymbol{u}, \quad (1)$$

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with the incompressibility condition

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

where p is the pressure, $\boldsymbol{u}$ is the velocity, and μ is the shear viscosity of the fluid, Let us consider spherical particles in a linear flow $\boldsymbol{u}^\infty$ given by

$$\boldsymbol{u}^\infty(\boldsymbol{x}) = \boldsymbol{U}^\infty + \boldsymbol{\Omega}^\infty \times \boldsymbol{x} + \boldsymbol{E}^\infty \cdot \boldsymbol{x}, \tag{3}$$

where the three constants $\boldsymbol{U}^\infty$, $\boldsymbol{\Omega}^\infty$, and $\boldsymbol{E}^\infty$ are the rigid translational velocity, rigid rotational velocity, and rate of strain of the imposed flow, respectively. According to the linearity of the Stokes equation, dynamics of the particles is completely characterized by the resistance equation (or, equivalently, the mobility equation, that is, the inverse of the resistance equation). For two-body problem, the equation is given [in (J-2)] by

$$\begin{bmatrix} \boldsymbol{F}^{(1)} \\ \boldsymbol{F}^{(2)} \\ \boldsymbol{T}^{(1)} \\ \boldsymbol{T}^{(2)} \\ \boldsymbol{S}^{(1)} \\ \boldsymbol{S}^{(2)} \end{bmatrix} = \mu \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \widetilde{\mathsf{B}}_{11} & \widetilde{\mathsf{B}}_{12} & \widetilde{\mathsf{G}}_{11} & \widetilde{\mathsf{G}}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \widetilde{\mathsf{B}}_{21} & \widetilde{\mathsf{B}}_{22} & \widetilde{\mathsf{G}}_{21} & \widetilde{\mathsf{G}}_{22} \\ \mathsf{B}_{11} & \mathsf{B}_{12} & \mathsf{C}_{11} & \mathsf{C}_{12} & \widetilde{\mathsf{H}}_{11} & \widetilde{\mathsf{H}}_{12} \\ \mathsf{B}_{21} & \mathsf{B}_{22} & \mathsf{C}_{21} & \mathsf{C}_{22} & \widetilde{\mathsf{H}}_{21} & \widetilde{\mathsf{H}}_{22} \\ \mathsf{G}_{11} & \mathsf{G}_{12} & \mathsf{H}_{11} & \mathsf{H}_{12} & \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{G}_{21} & \mathsf{G}_{22} & \mathsf{H}_{21} & \mathsf{H}_{22} & \mathsf{M}_{21} & \mathsf{M}_{22} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{U}^{(1)} - \boldsymbol{u}^\infty(\boldsymbol{x}_1) \\ \boldsymbol{U}^{(2)} - \boldsymbol{u}^\infty(\boldsymbol{x}_2) \\ \boldsymbol{\Omega}^{(1)} - \boldsymbol{\Omega}^\infty \\ \boldsymbol{\Omega}^{(2)} - \boldsymbol{\Omega}^\infty \\ \boldsymbol{E}^{(1)} - \boldsymbol{E}^\infty \\ \boldsymbol{E}^{(2)} - \boldsymbol{E}^\infty \end{bmatrix}, \tag{4}$$

where $\boldsymbol{F}^{(\alpha)}$, $\boldsymbol{T}^{(\alpha)}$, and $\boldsymbol{S}^{(\alpha)}$ are the force, torque, and stresslet of the particle α, and $\boldsymbol{U}^{(\alpha)}$, $\boldsymbol{\Omega}^{(\alpha)}$, and $\boldsymbol{E}^{(\alpha)}$ are the translational and angular velocities and strain of the particle α, respectively. In the equation, the grand resistance matrix is decomposed into 6×6 submatrices. Because of the symmetry of the grand resistance matrix, the matrices with tilde are obtained from the counterparts as $\widetilde{\mathsf{B}}_{\alpha\beta} = \mathsf{B}^\dagger_{\beta\alpha}$, $\widetilde{\mathsf{G}}_{\alpha\beta} = \mathsf{G}^\dagger_{\beta\alpha}$, and $\widetilde{\mathsf{H}}_{\alpha\beta} = \mathsf{H}^\dagger_{\beta\alpha}$, (where † denotes the transpose) and, therefore, we need to calculate, at least, the rest. Following Jeffrey *et al.*,[13,14] we scale these submatrices [in (JO-1.7a,b,c) and (J-3a,b,c)] as

$$\mathsf{A}_{\alpha\beta} = 3\pi\left(a_\alpha + a_\beta\right)\widehat{\mathsf{A}}_{\alpha\beta}, \tag{5a}$$
$$\mathsf{B}_{\alpha\beta} = \pi\left(a_\alpha + a_\beta\right)^2\widehat{\mathsf{B}}_{\alpha\beta}, \tag{5b}$$
$$\mathsf{C}_{\alpha\beta} = \pi\left(a_\alpha + a_\beta\right)^3\widehat{\mathsf{C}}_{\alpha\beta}, \tag{5c}$$
$$\mathsf{G}_{\alpha\beta} = \pi\left(a_\alpha + a_\beta\right)^2\widehat{\mathsf{G}}_{\alpha\beta}, \tag{5d}$$
$$\mathsf{H}_{\alpha\beta} = \pi\left(a_\alpha + a_\beta\right)^3\widehat{\mathsf{H}}_{\alpha\beta}, \tag{5e}$$
$$\mathsf{M}_{\alpha\beta} = \frac{5\pi}{6}\left(a_\alpha + a_\beta\right)^3\widehat{\mathsf{M}}_{\alpha\beta}, \tag{5f}$$

where a_α is the radius of particle α, and the matrices with hat are dimensionless. From the fact that the geometry of the problem is completely characterized by the single vector $\boldsymbol{r} = \boldsymbol{x}_\beta - \boldsymbol{x}_\alpha$, these submatrices can be given by scalar functions [in (JO-16a,b,c) and (J-4a,b,c)] as

$$\widehat{A}^{\alpha\beta}_{ij} = X^A_{\alpha\beta} e_i e_j + Y^A_{\alpha\beta}\left(\delta_{ij} - e_i e_j\right), \tag{6a}$$
$$\widehat{B}^{\alpha\beta}_{ij} = Y^B_{\alpha\beta}\epsilon_{ijk} e_k, \tag{6b}$$
$$\widehat{C}^{\alpha\beta}_{ij} = X^C_{\alpha\beta} e_i e_j + Y^C_{\alpha\beta}\left(\delta_{ij} - e_i e_j\right), \tag{6c}$$
$$\widehat{G}^{\alpha\beta}_{ijk} = X^G_{\alpha\beta}\left(e_i e_j - \frac{1}{3}\delta_{ij}\right)e_k + Y^G_{\alpha\beta}\left(e_i\delta_{jk} + e_j\delta_{ik} - 2e_i e_j e_k\right), \tag{6d}$$
$$\widehat{H}^{\alpha\beta}_{ijk} = Y^H_{\alpha\beta}\left(e_i\epsilon_{jkl}e_l + e_j\epsilon_{ikl}e_l\right), \tag{6e}$$
$$\begin{aligned}\widehat{M}^{\alpha\beta}_{ijkl} = {} & \frac{3}{2}X^M_{\alpha\beta}\left(e_i e_j - \frac{\delta_{ij}}{3}\right)\left(e_k e_l - \frac{\delta_{kl}}{3}\right) \\ & + \frac{Y^M_{\alpha\beta}}{2}\left(e_i\delta_{jl}e_k + e_j\delta_{il}e_k + e_i\delta_{jk}e_l + e_j\delta_{ik}e_l - 4e_i e_j e_k e_l\right) \\ & + \frac{Z^M_{\alpha\beta}}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl} + e_i e_j\delta_{kl} + \delta_{ij}e_k e_l + e_i e_j e_k e_l - e_i\delta_{jl}e_k - e_j\delta_{il}e_k - e_i\delta_{jk}e_l - e_j\delta_{ik}e_l\right),\end{aligned} \tag{6f}$$

where $\boldsymbol{e} = \boldsymbol{r}/|\boldsymbol{r}|$. The scalar functions such as $X^A_{\alpha\beta}$ and $Y^A_{\alpha\beta}$ are called the resistance functions. We have 11 functions for each pair $\alpha\beta$. From the symmetry on the exchange of particle indices α and β, we have the relations [in (JO-19a) – (JO-19e) and (J-5a) – (J-5f)] as

$$X^A_{\alpha\beta}(s,\lambda) = X^A_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7a}$$
$$Y^A_{\alpha\beta}(s,\lambda) = Y^A_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7b}$$
$$Y^B_{\alpha\beta}(s,\lambda) = -Y^B_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7c}$$
$$X^C_{\alpha\beta}(s,\lambda) = X^C_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7d}$$
$$Y^C_{\alpha\beta}(s,\lambda) = Y^C_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7e}$$
$$X^G_{\alpha\beta}(s,\lambda) = -X^G_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7f}$$
$$Y^G_{\alpha\beta}(s,\lambda) = -Y^G_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7g}$$
$$Y^H_{\alpha\beta}(s,\lambda) = Y^H_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7h}$$
$$X^M_{\alpha\beta}(s,\lambda) = X^M_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7i}$$
$$Y^M_{\alpha\beta}(s,\lambda) = Y^M_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7j}$$
$$Z^M_{\alpha\beta}(s,\lambda) = Z^M_{(3-\alpha)(3-\beta)}(s,\lambda^{-1}), \tag{7k}$$

where

$$s = \frac{2r}{a_1 + a_2}, \quad \lambda = \frac{a_2}{a_1}, \tag{8}$$

and $r = |\boldsymbol{r}|$. Therefore, once we have obtained 22 resistance functions for the particles 11 and 12, we can construct the grand resistance matrix completely. We will see the calculations in Sec IV.

B. Lamb's General Solution

In this paper, we utilize Lamb's general solution[17,18] to solve the problem. Lamb's general solution in the exterior region for the pressure and velocity fields p and $\boldsymbol{u}$ are given by

$$p(\boldsymbol{r}) = \sum_{n=0}^{\infty} p_{-n-1}, \tag{9}$$

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{r}) := \boldsymbol{u}(\boldsymbol{r}) - \boldsymbol{u}^{\infty} &= \sum_{n=0}^{\infty} \{\nabla \times (\boldsymbol{r}\chi_{-n-1}) + \nabla\Phi_{-n-1}\} \\ &+ \frac{1}{\mu}\sum_{n=1}^{\infty}\left\{-\frac{n-2}{2n(2n-1)} r^2 \nabla \frac{p_{-n-1}}{\mu}\right. \\ &\left. + \frac{n+1}{n(2n-1)} \boldsymbol{r} \frac{p_{-n-1}}{\mu}\right\}, \end{aligned} \tag{10}$$

where $\boldsymbol{u}^{\infty}$ is the imposed velocity and $\boldsymbol{v}$ is the disturbance velocity field. The solid spherical harmonics p_{-n-1}, Φ_{-n-1}, and χ_{-n-1} are expressed [in (JO-2.3)] by

$$\frac{p_{-n-1}}{\mu} = \sum_{m=0}^{n} p_{mn} \frac{1}{a} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta,\phi), \tag{11a}$$

$$\chi_{-n-1} = \sum_{m=0}^{n} q_{mn} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta,\phi), \tag{11b}$$

$$\Phi_{-n-1} = \sum_{m=0}^{n} v_{mn} a \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta,\phi), \tag{11c}$$

where Y_{mn} is the spherical harmonics defined by

$$Y_{mn}(\theta,\phi) = P_n^m(\cos\theta) e^{im\phi}, \tag{12}$$

with the associated Legendre function P_n^m, and p_{mn}, q_{mn}, and v_{mn} are the coefficients to be determined from the boundary conditions.

III. SINGLE SPHERE

First, let us consider a single sphere with radius a at the origin. On the particle surface $|\boldsymbol{r}| = a$, the conventional no-slip boundary condition is given by

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{E} \cdot \boldsymbol{r}, \tag{13}$$

where $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are the translational and rotational velocities of the particle, respectively. Here, we also introduce the strain tensor $\boldsymbol{E}$ of the particle surface, so that the boundary condition (13) is applicable to the deformable particle at instance of spherical shape. For rigid spherical particle, $\boldsymbol{E} = \boldsymbol{0}$.

A. Navier's Boundary Condition

Navier[2] proposed the slip boundary condition, where the slip velocity on the surface is proportional to the tangential force density, as

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{E} \cdot \boldsymbol{r} + \frac{\gamma}{\mu} (\boldsymbol{I} - \boldsymbol{nn}) \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n}), \tag{14}$$

where γ is the slip length, $\boldsymbol{I}$ is the unit tensor, n is the surface normal (equal to $\boldsymbol{r}/r$ for sphere), and $\boldsymbol{\sigma}$ is the stress tensor defined by

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\dagger}\right]. \tag{15}$$

Rewriting Eq. (14) by using the disturbance field $\boldsymbol{v}$ and imposed flow $\boldsymbol{u}^{\infty}$, we have

$$\boldsymbol{v} - \frac{\gamma}{\mu}\boldsymbol{t} = \boldsymbol{w}^{\Delta} + \frac{\gamma}{\mu}\boldsymbol{t}^{\infty}, \tag{16}$$

where

$$\boldsymbol{w}^{\Delta} = \Delta\boldsymbol{U} + \Delta\boldsymbol{\Omega} \times \boldsymbol{r} + \Delta\boldsymbol{E} \cdot \boldsymbol{r}, \tag{17}$$

and $\Delta\boldsymbol{U} = \boldsymbol{U} - \boldsymbol{U}^{\infty}$, $\Delta\boldsymbol{\Omega} = \boldsymbol{\Omega} - \boldsymbol{\Omega}^{\infty}$, and $\Delta\boldsymbol{E} = \boldsymbol{E} - \boldsymbol{E}^{\infty}$. The disturbance part $\boldsymbol{t}$ and imposed part $\boldsymbol{t}^{\infty}$ of the tangential force density are defined by

$$\boldsymbol{t} = (\boldsymbol{I} - \boldsymbol{nn}) \cdot (\boldsymbol{\sigma}^{v} \cdot \boldsymbol{n}), \tag{18a}$$

$$\boldsymbol{t}^{\infty} = (\boldsymbol{I} - \boldsymbol{nn}) \cdot (\boldsymbol{\sigma}^{\infty} \cdot \boldsymbol{n}), \tag{18b}$$

where

$$\boldsymbol{\sigma}^{v} = -p\boldsymbol{I} + \mu\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\dagger}\right], \tag{19a}$$

$$\boldsymbol{\sigma}^{\infty} = \mu\left[\nabla \boldsymbol{u}^{\infty} + (\nabla \boldsymbol{u}^{\infty})^{\dagger}\right]. \tag{19b}$$

From the imposed flow in Eq. (3), $\boldsymbol{t}^{\infty}$ becomes

$$\boldsymbol{t}^{\infty} = \frac{2\mu}{r} (\boldsymbol{I} - \boldsymbol{nn}) \cdot \boldsymbol{E}^{\infty} \cdot \boldsymbol{r}. \tag{20}$$

Note that, on the slip boundary condition (16), the left-hand side is the disturbance quantities and the right-hand side is the imposed quantities. Also note that, on the imposed part, the slip contribution appears only on the flow of $\boldsymbol{E}^{\infty} \neq \boldsymbol{0}$ as shown in Eq. (20).

In terms of Lamb's general solution for the disturbance field $\boldsymbol{v}$ in Eq. (10), the corresponding surface force density $\boldsymbol{f}$ is given by[17,18]

$$\begin{aligned} \boldsymbol{f} &:= \boldsymbol{\sigma}^{v} \cdot \boldsymbol{n} \\ &= \frac{\mu}{r}\sum_{n}\left\{-(n+2)\nabla \times (\boldsymbol{r}\chi_{-n-1})\right. \\ &\quad -2(n+2)\nabla\Phi_{-n-1} \\ &\quad \left. + \frac{1}{\mu}\frac{(n+1)(n-1)}{n(2n-1)} r^2 \nabla p_{-n-1} - \frac{1}{\mu}\frac{2n^2+1}{n(2n-1)} \boldsymbol{r} p_{-n-1}\right\}, \end{aligned} \tag{21}$$

and $\boldsymbol{t}$ defined in Eq. (18a) is expressed by

$$\begin{aligned} \boldsymbol{t} &= \frac{\mu}{r}\sum_{n}\left\{-(n+2)\nabla \times (\boldsymbol{r}\chi_{-n-1})\right. \\ &\quad -2(n+2)\left(\nabla - \frac{\boldsymbol{r}}{r}\partial_r\right)\Phi_{-n-1} \\ &\quad \left. + \frac{1}{\mu}\frac{(n+1)(n-1)}{n(2n-1)} r^2 \left(\nabla - \frac{\boldsymbol{r}}{r}\partial_r\right) p_{-n-1}\right\}. \end{aligned} \tag{22}$$

1. Three Scalar Functions

In order to achieve the boundary condition for Lamb's general solutions, Jeffrey and Onishi[13] used three scalar functions as in Happel and Brenner,[18] §3.2. Consider a general vector field $\boldsymbol{g}$ and its surface vectors $\boldsymbol{G}$ defined by

$$\boldsymbol{G}(\theta,\phi) := \boldsymbol{g}\Big|_{|\boldsymbol{r}|=a}, \tag{23}$$

so that

$$\frac{\partial \boldsymbol{G}}{\partial r} \equiv \boldsymbol{0}. \tag{24}$$

We define the following three scalar functions

$$G_{\rm rad} := \frac{\boldsymbol{r}}{r}\cdot\boldsymbol{G}, \tag{25a}$$

$$G_{\rm div} := -r\boldsymbol{\nabla}\cdot\boldsymbol{G}, \tag{25b}$$

$$G_{\rm rot} := \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{G}. \tag{25c}$$

Obviously, the first scalar $G_{\rm rad}$ is the radial component $G_r = (\boldsymbol{r}/r)\cdot\boldsymbol{G}$ itself. The other two, $G_{\rm div}$ and $G_{\rm rot}$, are related to the tangential components (*i.e.*, G_θ and G_ϕ in polar coordinates), except for the factor $-2G_r$ on the divergence, as

$$G_{\rm div} = -2G_r - \left(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\right)G_\theta - \frac{1}{\sin\theta}\frac{\partial G_\phi}{\partial\phi}, \tag{26a}$$

$$G_{\rm rad} = \mathscr{S}\left[-\frac{1}{\sin\theta}\frac{\partial G_\theta}{\partial\phi} + \left(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\right)G_\phi\right], \tag{26b}$$

where $\mathscr{S}$ is $+1$ in the right-handed coordinates and -1 in the left-handed coordinates. It should be noted that the divergence of the surface vector $\boldsymbol{G}$ is related to the 3D vector field $\boldsymbol{g}$ as

$$G_{\rm div} = -r\boldsymbol{\nabla}\cdot\boldsymbol{g}\Big|_{|\boldsymbol{r}|=a} + r\frac{\partial}{\partial r}g_r\Big|_{|\boldsymbol{r}|=a}, \tag{27}$$

where the substitution of $|\boldsymbol{r}| = a$ is applied after the derivatives.

a. Velocity Field As a first example, consider the disturbance velocity $\boldsymbol{v}$, whose surface vector is defined by $\boldsymbol{V}$ as

$$\boldsymbol{V}(\theta,\phi) := \boldsymbol{v}\Big|_{|\boldsymbol{r}|=a}. \tag{28}$$

By definition, the first scalar $V_{\rm rad}$ is given by $\boldsymbol{v}$ as

$$V_{\rm rad} := \frac{\boldsymbol{r}}{r}\cdot\boldsymbol{V} = \frac{\boldsymbol{r}}{r}\cdot\boldsymbol{v}\Big|_{|\boldsymbol{r}|=a}. \tag{29a}$$

Because $\boldsymbol{v}$ satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, $V_{\rm div}$ is given by

$$V_{\rm div} := -r\boldsymbol{\nabla}\cdot\boldsymbol{V} = r\frac{\partial}{\partial r}v_r\Big|_{|\boldsymbol{r}|=a}, \tag{29b}$$

from Eq. (27). $V_{\rm rad}$ is independent of its radial component V_r as shown in Eq. (26b), so that it is simply written by $\boldsymbol{v}$ as

$$V_{\rm rot} := \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{V} = \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{v}\Big|_{|\boldsymbol{r}|=a}. \tag{29c}$$

From Lamb's general solution for $\boldsymbol{v}$ in Eq. (10), then, the three scalars are obtained as in Jeffrey and Onishi.[13]

b. Tangential Surface Force Next, let us consider $\boldsymbol{t}$ which is necessary for the slip boundary condition in Eq. (16). Its surface vector is defined by

$$\boldsymbol{T}(\theta,\phi) := \boldsymbol{t}\Big|_{|\boldsymbol{r}|=a}. \tag{30}$$

The radial component of $\boldsymbol{t}$ is zero by definition as

$$T_{\rm rad} := \frac{\boldsymbol{r}}{r}\cdot\boldsymbol{T} = 0. \tag{31a}$$

From Eq. (27), therefore, we have

$$T_{\rm div} := -r\boldsymbol{\nabla}\cdot\boldsymbol{T} = -r\boldsymbol{\nabla}\cdot\boldsymbol{t}\Big|_{|\boldsymbol{r}|=a}. \tag{31b}$$

Because the rotation has no radial component for an arbitrary vector field, we can use the bare surface force $\boldsymbol{f}$ for the boundary condition for the tangential force $\boldsymbol{t}$ as

$$T_{\rm rot} := \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{T} = \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{t}\Big|_{|\boldsymbol{r}|=a} = \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{f}\Big|_{|\boldsymbol{r}|=a}. \tag{31c}$$

Using Lamb's general solution in Eq. (22), the three scalar components for $\boldsymbol{t}$ are given by

$$\frac{r_i}{r}t_i = 0, \tag{32a}$$

$$-r\boldsymbol{\nabla}\cdot\boldsymbol{t} = -\mu\sum_n\left[\frac{2n(n+1)(n+2)}{r^2}\Phi_{-n-1} - \frac{(n+1)^2(n-1)}{2n-1}\frac{p_{-n-1}}{\mu}\right], \tag{32b}$$

$$\boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{t} = -\frac{\mu}{r}\sum_n(n+2)n(n+1)\chi_{-n-1}. \tag{32c}$$

c. Disturbance Part Three scalars for $\boldsymbol{V}$ are obtained by Eqs. (29a), (29b), and (29c), and the slip contribution $-(\gamma/\mu)\boldsymbol{T}$ by Eqs. (32a), (32b), and (32c). Substituting Lamb's solution (10) with the expansions in Eqs. (11a), (11b), and (11c) and putting $r = a$, the three scalars of the disturbance part, *i.e.*, the left-hand side, of the slip boundary condition (16) are given by

$$\left(\boldsymbol{V} - \frac{\gamma}{\mu}\boldsymbol{T}\right)_{\rm rad} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[-(n+1)v_{mn} + \frac{n+1}{2(2n-1)}p_{mn}\right]Y_{mn}(\theta,\phi), \tag{33a}$$

$$\left(\boldsymbol{V} - \frac{\gamma}{\mu}\boldsymbol{T}\right)_{\rm div} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[(n+1)(n+2)\left(1+2n\widehat{\gamma}\right)v_{mn} - \frac{n(n+1)}{2(2n-1)}\left(1+\frac{2(n+1)(n-1)}{n}\widehat{\gamma}\right)p_{mn}\right] Y_{mn}(\theta,\phi), \tag{33b}$$

$$\left(\boldsymbol{V} - \frac{\gamma}{\mu}\boldsymbol{T}\right)_{\rm rot} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} n(n+1)\left(1+(n+2)\widehat{\gamma}\right)q_{mn}Y_{mn}(\theta,\phi), \tag{33c}$$

where the scaled slip length $\widehat{\gamma}$ is defined by

$$\widehat{\gamma} := \frac{\gamma}{a}. \tag{34}$$

d. Imposed Part Let us look at the three components for the vector $\boldsymbol{w}^\Delta$ in Eq. (17). Note that the divergence is zero as shown by

$$\partial_i w_i^\Delta = \epsilon_{ijk}\Delta\Omega_j\delta_{ik} + \Delta E_{ij}\delta_{ij} = 0, \tag{35}$$

because $E_{kk} = 0$. Therefore, we need to calculate the divergence component through the derivative of the radial velocity (as for $\boldsymbol{v}$). The three components for $\boldsymbol{w}^\Delta$ are then given by

$$\frac{r_i}{r}w_i^\Delta = \frac{r_i}{r}\Delta U_i + \frac{r_i r_j}{r}\Delta E_{ij}, \tag{36a}$$

$$r_j\partial_j\frac{r_i}{r}w_i^\Delta = \frac{r_i r_j}{r}\Delta E_{ij}, \tag{36b}$$

$$r_i\epsilon_{ijk}\partial_j w_k^\Delta = 2r_i\Delta\Omega_i. \tag{36c}$$

We use the identity $\epsilon_{ijk}\epsilon_{jkl} = 2\delta_{il}$ for the last equation. For $\boldsymbol{t}^\infty$, the three components are given as in the following. The normal component is zero by definition as

$$\frac{r_i}{r}t_i^\infty = 0. \tag{37}$$

The rotation is also vanished as

$$r_i\epsilon_{ijk}\partial_j t_k^\infty = 2\mu r_i\epsilon_{ijk}\partial_j\left(\delta_{kl}\frac{r_m}{r} - \frac{r_k r_l r_m}{r^3}\right)E_{lm}^\infty = 0. \tag{38}$$

Because the normal component is zero (as for $\boldsymbol{t}$), the divergence component is obtained through the divergence as

$$-r\partial_i t_k^\infty = -2\mu r\partial_i\left(\delta_{ij}\frac{r_k}{r} - \frac{r_i r_j r_k}{r^3}\right)E_{jk}^\infty = 6\mu\frac{r_j r_k}{r^2}E_{jk}^\infty, \tag{39}$$

where we use $E_{kk}^\infty = 0$.

Define the surface vector of the right-hand side of the slip boundary condition (16) by

$$\boldsymbol{W} := \left.\left(\boldsymbol{w}^\Delta + \frac{\gamma}{\mu}\boldsymbol{t}^\infty\right)\right|_{|\boldsymbol{r}|=a}. \tag{40}$$

The three scalars for $\boldsymbol{W}$ are then given by

$$W_{\rm rad} = e_i\Delta U_i + e_i e_j a\Delta E_{ij}, \tag{41a}$$

$$W_{\rm div} = e_i e_j a\Delta E_{ij} + 6\widehat{\gamma} e_i e_j a E_{ij}^\infty, \tag{41b}$$

$$W_{\rm rot} = 2e_i a\Delta\Omega_i, \tag{41c}$$

where $\boldsymbol{e} = \boldsymbol{r}/r$.

2. Recurrence Relations

Let us introduce the spherical harmonics expansion for the three components of the imposed part by

$$W_{\rm rad} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\chi_{mn}Y_{mn}(\theta,\phi), \tag{42a}$$

$$W_{\rm div} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\psi_{mn}Y_{mn}(\theta,\phi), \tag{42b}$$

$$W_{\rm rot} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\omega_{mn}Y_{mn}(\theta,\phi). \tag{42c}$$

From Eqs. (41a), (41b), and (41c), the coefficients χ_{mn}, ψ_{mn}, and ω_{mn} are given by the parameters $\Delta\boldsymbol{U}$, $\Delta\boldsymbol{\Omega}$, $\Delta\boldsymbol{E}$, and $\boldsymbol{E}^\infty$. Therefore, by the boundary condition (16) at the surface $|\boldsymbol{r}| = a$ with the scalars of the disturbance fields in Eqs. (33a), (33b), and (33c), the coefficients (p_{mn}, q_{mn}, v_{mn}) are given by the boundary condition $(\chi_{mn}, \psi_{mn}, \omega_{mn})$ as

$$p_{mn} = \frac{2n-1}{n+1}\Gamma_{0,2n+1}\psi_{mn} + \frac{(n+2)(2n-1)}{n+1}\Gamma_{2n,2n+1}\chi_{mn}, \tag{43a}$$

$$v_{mn} = \frac{1}{2(n+1)}\Gamma_{0,2n+1}\psi_{mn} + \frac{n}{2(n+1)}\Gamma_{2(n+1)(n-1)/n,2n+1}\chi_{mn}, \tag{43b}$$

$$q_{mn} = \frac{1}{n(n+1)}\Gamma_{0,n+2}\omega_{mn}, \tag{43c}$$

where

$$\Gamma_{m,n} = \frac{1+m\widehat{\gamma}}{1+n\widehat{\gamma}}. \tag{44}$$

Note that in the no-slip ($\widehat{\gamma} = 0$) and perfect-slip ($\widehat{\gamma} = \infty$) limits, $\Gamma_{m,n}$ reduces to

$$\Gamma_{m,n} = \begin{cases} 1 & \text{for } \widehat{\gamma} = 0, \\ m/n & \text{for } \widehat{\gamma} = \infty. \end{cases} \tag{45}$$

B. Single Body Solutions

In the following, we solve single-body problem with the slip boundary condition through Eqs. (43a), (43b), and (43c).

1. Translating Sphere

Consider the translating sphere with the velocity $\boldsymbol{U} = (0, 0, U)$, which is given by

$$\chi_{m,n} = U\delta_{0m}\delta_{1n}. \tag{46}$$

Substituting the condition (46) into the recurrence relations (43a), (43b), and (43c), we have the solution

$$p_{mn} = \frac{3}{2}U\Gamma_{2,3}\delta_{m0}\delta_{n1}, \tag{47a}$$

$$v_{mn} = \frac{1}{4}U\Gamma_{0,3}\delta_{m0}\delta_{n1}, \tag{47b}$$

$$q_{mn} = 0. \tag{47c}$$

The force acting on the particle is given by the coefficient of Lamb's general solution [in (JO-2.10)] as

$$\boldsymbol{F} = 4\pi\mu a\left[p_{01}\hat{\boldsymbol{z}} - p_{11}\left(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}\right)\right], \tag{48}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are the unit vectors in x, y, and z directions, respectively. Therefore, the force on the sphere translating with the velocity U in z direction is

$$\boldsymbol{F} = 6\pi\mu a\Gamma_{2,3}U\hat{\boldsymbol{z}}. \tag{49}$$

This is identical to the result by Basset.[7] (See also Lamb,[17] Art. 337, 3° and Felderhof.[8]) Substituting the coefficients (47a), (47b), and (47c) into Lamb's general solution in Eq. (10) and rewriting the parameter U by the strength of the force F through Eq. (49), the disturbance field is given by

$$\boldsymbol{v} = \frac{1}{8\pi\mu}\left(1 + \Gamma_{0,2}\frac{a^2}{6}\nabla^2\right)\boldsymbol{J}\cdot\boldsymbol{F}, \tag{50}$$

where $\boldsymbol{J}$ is the Oseen-Burgers tensor given by

$$J_{ij}(\boldsymbol{r}) = \frac{1}{r}\left(\delta_{ij} + \frac{r_i r_j}{r^2}\right). \tag{51}$$

2. Rotating Sphere

For the Ω problem, $W_{\rm rot}$ in Eq. (41c) is the only non-zero component. Consider the sphere with the angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$, which reduces to

$$\omega_{m,n} = 2a\Omega\delta_{0m}\delta_{1n}. \tag{52}$$

Substituting the condition (46) into the recurrence relations (43a), (43b), and (43c), we have the solution

$$p_{mn} = 0, \tag{53a}$$
$$v_{mn} = 0, \tag{53b}$$
$$q_{mn} = a\Omega\Gamma_{0,3}\delta_{m0}\delta_{n1}. \tag{53c}$$

The torque acting on the particle is given by the coefficient of Lamb's general solution [in (JO-2.11)] as

$$\boldsymbol{T} = 8\pi\mu a^2\left[q_{01}\hat{\boldsymbol{z}} - q_{11}\left(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}\right)\right]. \tag{54}$$

Therefore, the torque on the sphere rotating with the angular velocity Ω in z direction is

$$\boldsymbol{T} = 8\pi\mu a^3\Gamma_{0,3}\Omega\hat{\boldsymbol{z}}. \tag{55}$$

This is consistent with the result by Felderhof[8] and Padmavathi *et al.*[19] Note that the torque $\boldsymbol{T}$ would vanish for the sphere with the perfect-slip surface (for $\widehat{\gamma} = \infty$). Substituting the coefficients (53a), (53b), and (53c) into Lamb's general solution in Eq. (10) and using Eq. (55), the disturbance field is given by

$$\boldsymbol{v} = \frac{1}{8\pi\mu}\boldsymbol{R}\cdot\boldsymbol{T}, \tag{56}$$

where

$$R_{ij}(\boldsymbol{r}) := \epsilon_{ijk}\frac{r_k}{r^3}. \tag{57}$$

3. Sphere in Shear Flow

For the E problem, we have two non-zero components. Here we assume the rigid sphere, so that $\boldsymbol{E} = \boldsymbol{0}$ and from Eqs. (41a), (41b), and (41c),

$$W_{\rm rad} = -e_i e_j a E^\infty_{ij}, \tag{58a}$$
$$W_{\rm div} = -e_i e_j a E^\infty_{ij}\left(1 - 6\widehat{\gamma}\right), \tag{58b}$$
$$W_{\rm rot} = 0. \tag{58c}$$

Let us consider the strain given by

$$-E^\infty_{ij} = E\left(\hat{z}_i\hat{z}_j - \frac{1}{3}\delta_{ij}\right). \tag{59}$$

This is achieved by

$$\chi_{m,n} = \frac{2}{3}aE\delta_{0m}\delta_{2n}, \tag{60a}$$
$$\psi_{m,n} = \frac{2}{3}aE\left(1 - 6\widehat{\gamma}\right)\delta_{0m}\delta_{2n}. \tag{60b}$$

Substituting the boundary conditions (60a) and (60b) into the recurrence relations (43a), (43b), and (43c), we have the solution

$$p_{mn} = \frac{10}{3}aE\Gamma_{2,5}\delta_{m0}\delta_{n2}, \tag{61a}$$
$$v_{mn} = \frac{1}{3}aE\Gamma_{0,5}\delta_{m0}\delta_{n2}, \tag{61b}$$
$$q_{mn} = 0. \tag{61c}$$

The stresslet acting on the particle is given by the coefficient of Lamb's general solution [in (J-6)] as

$$\begin{aligned}\boldsymbol{S} = 2\pi\mu a^2\Big\{&p_{02}\left(\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} - \frac{1}{3}\boldsymbol{I}\right)\\ &-p_{12}\left[\hat{\boldsymbol{x}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \mathrm{i}\left(\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{y}}\right)\right]\\ &+2p_{22}\left[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \mathrm{i}\left(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{x}}\right)\right]\Big\}.\end{aligned} \tag{62}$$

Therefore, the stresslet on the sphere in the shear flow with the parameter E is

$$\boldsymbol{S} = \frac{20}{3}\pi\mu a^3\Gamma_{2,5}\boldsymbol{E}. \tag{63}$$

This is identical to the result by Felderhof.[8] Note that this has the following two extremes

$$\boldsymbol{S} = \begin{cases}\frac{20}{3}\pi\mu a^3\boldsymbol{E} & \text{for } \widehat{\gamma} = 0 \text{ (no-slip)},\\ \frac{8}{3}\pi\mu a^3\boldsymbol{E} & \text{for } \widehat{\gamma} = \infty \text{ (perfect-slip)}.\end{cases} \tag{64}$$

These yield to the effective viscosity μ^* up to $O(\phi)$ in the two limits as

$$\frac{\mu^*}{\mu} = \begin{cases}1 + \frac{5}{2}\phi & \text{for no-slip particles},\\ 1 + \phi & \text{for perfect-slip particles},\end{cases} \tag{65}$$

where ϕ is the volume fraction. From Batchelor,[20] the effective viscosity for the dispersions of fluid droplets is given by

$$\frac{\mu^*}{\mu} = 1 + \phi\left(\frac{\mu + \frac{5}{2}\overline{\mu}}{\mu + \overline{\mu}}\right), \tag{66}$$

where μ is the viscosity of the fluid surrounding the droplets and $\overline{\mu}$ is the viscosity of the fluid inside the droplets. This gives the same extremes as

$$\frac{\mu^*}{\mu} = \begin{cases}1 + \frac{5}{2}\phi & \text{for } \overline{\mu} = \infty \text{ (rigid particles)},\\ 1 + \phi & \text{for } \overline{\mu} = 0 \text{ (bubbles)},\end{cases} \tag{67}$$

as expected. Substituting the coefficients (61a), (61b), and (61c) into Lamb's general solution in Eq. (10) and using Eq. (63), the disturbance field is given by

$$\boldsymbol{v} = -\frac{1}{8\pi\mu}\left(1 + \Gamma_{0,2}\frac{a^2\nabla^2}{10}\right)\boldsymbol{K} : \boldsymbol{S}, \qquad (68)$$

where

$$K_{ijk}(\boldsymbol{r}) = -3\frac{r_i r_j r_k}{r^5}. \qquad (69)$$

IV. TWO-BODY PROBLEM

Now, we study two-body problem. We will determine 22 resistance functions mentioned in Sec. II A. Following Jeffrey *et al.*,[13,14] we write these functions in terms of the coefficients f_m and determine the coefficients. Here we summarize the definitions of the coefficients: $X^A_{\alpha\beta}$ are given [in (JO-3.13) and (JO-3.14)] by

$$X^A_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{XA}_m}{[(1+\lambda)s]^m}, \qquad (70a)$$

$$X^A_{12}(s,\lambda) = \frac{-2}{1+\lambda}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{XA}_m}{[(1+\lambda)s]^m}, \qquad (70b)$$

$Y^A_{\alpha\beta}$ [in (JO-4.13) and (JO-4.14)] by

$$Y^A_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{YA}_m}{[(1+\lambda)s]^m}, \qquad (71a)$$

$$Y^A_{12}(s,\lambda) = \frac{-2}{1+\lambda}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{YA}_m}{[(1+\lambda)s]^m}, \qquad (71b)$$

$Y^B_{\alpha\beta}$ [in (JO-5.3) and (JO-5.4)] by

$$Y^B_{11}(s,\lambda) = \sum_{m=1,\text{odd}}^{\infty} \frac{f^{YB}_m}{[(1+\lambda)s]^m}, \qquad (72a)$$

$$Y^B_{12}(s,\lambda) = \frac{-4}{(1+\lambda)^2}\sum_{m=0,\text{even}}^{\infty} \frac{f^{YB}_m}{[(1+\lambda)s]^m}, \qquad (72b)$$

$X^C_{\alpha\beta}$ [in (JO-6.7) and (JO-6.8)] by

$$X^C_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{XC}_m}{[(1+\lambda)s]^m}, \qquad (73a)$$

$$X^C_{12}(s,\lambda) = \frac{-8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{XC}_m}{[(1+\lambda)s]^m}, \qquad (73b)$$

$Y^C_{\alpha\beta}$ [in (JO-7.7) and (JO-7.8)] by

$$Y^C_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{YC}_m}{[(1+\lambda)s]^m}, \qquad (74a)$$

$$Y^C_{12}(s,\lambda) = \frac{8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{YC}_m}{[(1+\lambda)s]^m}, \qquad (74b)$$

$X^G_{\alpha\beta}$ [in (J-18a,b)] by

$$X^G_{11}(s,\lambda) = \sum_{m=1,\text{odd}}^{\infty} \frac{f^{XG}_m}{[(1+\lambda)s]^m}, \qquad (75a)$$

$$X^G_{12}(s,\lambda) = \frac{-4}{(1+\lambda)^2}\sum_{m=2,\text{even}}^{\infty} \frac{f^{XG}_m}{[(1+\lambda)s]^m}, \qquad (75b)$$

$Y^G_{\alpha\beta}$ [in (J-26a,b)] by

$$Y^G_{11}(s,\lambda) = \sum_{m=1,\text{odd}}^{\infty} \frac{f^{YG}_m}{[(1+\lambda)s]^m}, \qquad (76a)$$

$$Y^G_{12}(s,\lambda) = \frac{-4}{(1+\lambda)^2}\sum_{m=0,\text{even}}^{\infty} \frac{f^{YG}_m}{[(1+\lambda)s]^m}, \qquad (76b)$$

$Y^H_{\alpha\beta}$ [in (J-34a,b)] by

$$Y^H_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{YH}_m}{[(1+\lambda)s]^m}, \qquad (77a)$$

$$Y^H_{12}(s,\lambda) = \frac{8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{YH}_m}{[(1+\lambda)s]^m}, \qquad (77b)$$

$X^M_{\alpha\beta}$ [in (J-47a,b)] by

$$X^M_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{XM}_m}{[(1+\lambda)s]^m}, \qquad (78a)$$

$$X^M_{12}(s,\lambda) = \frac{8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{XM}_m}{[(1+\lambda)s]^m}, \qquad (78b)$$

$Y^M_{\alpha\beta}$ [in (J-63a,b)] by

$$Y^M_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{YM}_m}{[(1+\lambda)s]^m}, \qquad (79a)$$

$$Y^M_{12}(s,\lambda) = \frac{8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{YM}_m}{[(1+\lambda)s]^m}, \qquad (79b)$$

$Z^M_{\alpha\beta}$ [in (J-78a,b)] by

$$Z^M_{11}(s,\lambda) = \sum_{m=0,\text{even}}^{\infty} \frac{f^{ZM}_m}{[(1+\lambda)s]^m}, \qquad (80a)$$

$$Z^M_{12}(s,\lambda) = \frac{-8}{(1+\lambda)^3}\sum_{m=1,\text{odd}}^{\infty} \frac{f^{ZM}_m}{[(1+\lambda)s]^m}. \qquad (80b)$$

Before proceeding to the full calculation of the coefficients by twin multipole expansions (shown later in Sec IV B), we first derive lower coefficients by simpler formulation called the method of reflections.

A. Method of Reflections

Let us consider two particles $\alpha = 1$ and 2, whose centers, radii, and slip lengths are given by $\boldsymbol{x}_\alpha$, a_α, and γ_α, respectively. The scaled slip length for particle α is defined by

$$\widehat{\gamma}_\alpha := \frac{\gamma_\alpha}{a_\alpha}. \tag{81}$$

1. Faxén's Laws

From Eqs. (50), (56), and (68) in the previous section, the disturbance velocity field at position $\boldsymbol{x}$ caused by a single sphere α at $\boldsymbol{x}_\alpha$ with slip length γ_α is given by

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) &= \frac{1}{8\pi\mu}\left[\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2}{6}\nabla^2\right)\boldsymbol{J}(\boldsymbol{x}-\boldsymbol{x}_\alpha)\cdot\boldsymbol{F}^{(\alpha)}\right. \\ &\quad +\boldsymbol{R}(\boldsymbol{x}-\boldsymbol{x}_\alpha)\cdot\boldsymbol{T}^{(\alpha)} \\ &\quad \left. -\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2\nabla^2}{10}\right)\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{x}_\alpha):\boldsymbol{S}^{(\alpha)}\right], \end{aligned} \tag{82}$$

where

$$\Gamma^{(\alpha)}_{m,n} := \frac{1+m\widehat{\gamma}_\alpha}{1+n\widehat{\gamma}_\alpha}, \tag{83}$$

and the force $\boldsymbol{F}^{(\alpha)}$, torque $\boldsymbol{T}^{(\alpha)}$, and stresslet $\boldsymbol{S}^{(\alpha)}$ on the sphere are given by

$$\boldsymbol{F}^{(\alpha)} = 6\pi\mu a_\alpha\Gamma^{(\alpha)}_{2,3}\boldsymbol{U}^{(\alpha)}, \tag{84a}$$

$$\boldsymbol{T}^{(\alpha)} = 8\pi\mu a_\alpha^3\Gamma^{(\alpha)}_{0,3}\boldsymbol{\Omega}^{(\alpha)}, \tag{84b}$$

$$\boldsymbol{S}^{(\alpha)} = \frac{20}{3}\pi\mu a_\alpha^3\Gamma^{(\alpha)}_{2,5}\boldsymbol{E}^{(\alpha)}. \tag{84c}$$

(See Eqs. (49), (55), and (63) in the previous section.) Reading Eq. (82) as multipole expansion of the velocity field, Faxén's laws for slip sphere are derived as

$$\boldsymbol{F}^{(\alpha)} = 6\pi\mu a_\alpha\Gamma^{(\alpha)}_{2,3}\left[\boldsymbol{U}^{(\alpha)}-\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2}{6}\nabla^2\right)\boldsymbol{u}'(\boldsymbol{x}_\alpha)\right], \tag{85}$$

$$\boldsymbol{T}^{(\alpha)} = 8\pi\mu a_\alpha^3\Gamma^{(\alpha)}_{0,3}\left[\boldsymbol{\Omega}^{(\alpha)}-\frac{1}{2}\left(\boldsymbol{\nabla}\times\boldsymbol{u}'\right)(\boldsymbol{x}_\alpha)\right], \tag{86}$$

$$\begin{aligned} \boldsymbol{S}^{(\alpha)} &= \frac{20}{3}\pi\mu a_\alpha^3\Gamma^{(\alpha)}_{2,5}\left[\boldsymbol{E}^{(\alpha)}\right. \\ &\quad \left.-\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2\nabla^2}{10}\right)\frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}'+(\boldsymbol{\nabla}\boldsymbol{u})^\dagger\right)(\boldsymbol{x}_\alpha)\right], \end{aligned} \tag{87}$$

where $\boldsymbol{u}'$ is the velocity field in absent of particle α. For later use, we rewrite Eq. (82) in the resistance form by replacing $\boldsymbol{F}^{(\alpha)}$, $\boldsymbol{T}^{(\alpha)}$, and $\boldsymbol{S}^{(\alpha)}$ by $\boldsymbol{U}^{(\alpha)}$, $\boldsymbol{\Omega}^{(\alpha)}$, and $\boldsymbol{E}^{(\alpha)}$ from Eqs. (84a), (84b), and (84c) as

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) &= \frac{3a_\alpha}{4}\Gamma^{(\alpha)}_{2,3}\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2}{6}\nabla^2\right)\boldsymbol{J}(\boldsymbol{x}-\boldsymbol{x}_\alpha)\cdot\boldsymbol{U}^{(\alpha)} \\ &\quad +a_\alpha^3\Gamma^{(\alpha)}_{0,3}\boldsymbol{R}(\boldsymbol{x}-\boldsymbol{x}_\alpha)\cdot\boldsymbol{\Omega}^{(\alpha)} \\ &\quad -\frac{5a_\alpha^3}{6}\Gamma^{(\alpha)}_{2,5}\left(1+\Gamma^{(\alpha)}_{0,2}\frac{a_\alpha^2\nabla^2}{10}\right)\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{x}_\alpha):\boldsymbol{E}^{(\alpha)}. \end{aligned} \tag{88}$$

2. Translating Spheres in Axisymmetric Motion

Here we set the relative vector between particle 1 and 2 in z direction as

$$\boldsymbol{r} := \boldsymbol{x}_2 - \boldsymbol{x}_1 = (0,0,r). \tag{89}$$

For the function X^A, we set the velocity of the particle 1 parallel to $\boldsymbol{r}$ as

$$\boldsymbol{U}^{(1)} = (0,0,U^{(1)}). \tag{90}$$

From Faxén's law for the force in Eq. (85) with the disturbance field by Eq. (88) with Eq. (90), we have the force on the particle 2 due to the translating particle 1 as

$$\begin{aligned} F_i^{(2)} &= 6\pi\mu a_2\Gamma^{(2)}_{2,3}U_i^{(2)} \\ &\quad -6\pi\mu a_2\left[\frac{3}{2}\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,3}\frac{a_1}{r}-\frac{1}{2}\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{0,3}\frac{a_1^3}{r^3}\right. \\ &\quad \left.-\frac{1}{2}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,3}\frac{a_1a_2^2}{r^3}\right]U^{(1)}\delta_{iz}. \end{aligned} \tag{91}$$

In terms of the scalar functions $X^A_{\alpha\beta}$, the force is expressed by

$$\begin{aligned} F_i^{(2)} &= 6\pi\mu a_2X^A_{22}(s,\lambda)U^{(2)}\delta_{iz} \\ &\quad +3\pi\mu(a_2+a_1)X^A_{21}(s,\lambda)U^{(1)}\delta_{iz}, \end{aligned} \tag{92}$$

where s and λ are defined in Eq. (8). Therefore,

$$X^A_{22}(s,\lambda) = \Gamma^{(2)}_{2,3}, \tag{93a}$$

$$X^A_{21}(s,\lambda) = \frac{-2\lambda}{1+\lambda}\left(\frac{3\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,3}}{(1+\lambda)s}-\frac{4\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{0,3}+4\lambda^2\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,3}}{(1+\lambda)^3s^3}\right). \tag{93b}$$

From the symmetry of $X^A_{\alpha\beta}$ in Eq. (7a), we have

$$X^A_{12}(s,\lambda) = \frac{-2}{1+\lambda}\left(\frac{3\lambda\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}}{(1+\lambda)s}-\frac{4\lambda^3\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{0,3}+4\lambda\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,3}}{(1+\lambda)^3s^3}\right). \tag{94}$$

From the expression of X^A_{12} by the coefficients f_k^{XA} in Eq. (70b), we have

$$f_1^{XA} = 3\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\lambda, \tag{95a}$$

$$f_3^{XA} = -4\lambda\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,3}-4\lambda^3\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{0,3}. \tag{95b}$$

For the self part X^A_{11}, we have

$$f_0^{XA} = \Gamma^{(1)}_{2,3}. \tag{95c}$$

These coefficients (and those for the rest of the functions below) will be compared with the results by twin multipole expansions in Sec. IV B.

From Faxén's law for the torque in Eq. (86), we have the torque on the particle 2 due to the translating particle 1 as

$$T_i^{(2)} = 0, \tag{96}$$

because $\mathbf{\Omega}^\alpha = 0$ in the present problem and $\partial_j u_k^{(1)}$ is symmetric about the indices j, k. This is consistent that there is no X^B function.

From Faxén's law for the stresslet in Eq. (87),

$$S_{ij}^{(2)} = \frac{20}{3}\pi\mu a_2^3 \Gamma_{2,5}^{(2)} E_{ij}^{(2)} - \frac{20}{3}\pi\mu a_2^3 \left(-\frac{9}{4}\frac{a_1}{r^2}\Gamma_{2,5}^{(2)}\Gamma_{2,3}^{(1)} + \frac{9}{4}\frac{a_1^3}{r^4}\Gamma_{2,5}^{(2)}\Gamma_{0,3}^{(1)} + \frac{27 a_1 a_2^2}{20 r^4}\Gamma_{0,5}^{(2)}\Gamma_{2,3}^{(1)} \right) U^{(1)} \left(\delta_{iz}\delta_{jz} - \frac{\delta_{ij}}{3} \right). \tag{97}$$

In terms of the scalar functions $X_{\alpha\beta}^G$, the stresslet is expressed by

$$S_{ij}^{(2)} = \mu\pi (a_2 + a_1)^2 X_{21}^G U^{(1)} \left(\delta_{iz}\delta_{jz} - \frac{1}{3}\delta_{ij} \right), \tag{98}$$

so that

$$X_{21}^G = \frac{-4\lambda^3}{(1+\lambda)^2}\left[-\frac{15}{(1+\lambda)^2 s^2}\Gamma_{2,5}^{(2)}\Gamma_{2,3}^{(1)} + \frac{60}{(1+\lambda)^4 s^4}\Gamma_{2,5}^{(2)}\Gamma_{0,3}^{(1)} + \frac{36\lambda^2}{(1+\lambda)^4 s^4}\Gamma_{0,5}^{(2)}\Gamma_{2,3}^{(1)} \right]. \tag{99}$$

From the symmetry of $X_{\alpha\beta}^G$ in Eq. (7f), we have

$$X_{12}^G = \frac{-4}{(1+\lambda)^2}\left[\frac{15\lambda\Gamma_{2,5}^{(1)}\Gamma_{2,3}^{(2)}}{(1+\lambda)^2 s^2} - \frac{60\lambda^3\Gamma_{2,5}^{(1)}\Gamma_{0,3}^{(2)} + 36\lambda\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)}}{(1+\lambda)^4 s^4} \right]. \tag{100}$$

From the expression of X_{12}^G by the coefficients f_k^{XG} in Eq. (75b), we have

$$f_0^{XG} = 0, \tag{101a}$$
$$f_2^{XG} = 15\Gamma_{2,5}^{(1)}\Gamma_{2,3}^{(2)}\lambda, \tag{101b}$$
$$f_4^{XG} = -36\lambda\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)} - 60\lambda^3\Gamma_{2,5}^{(1)}\Gamma_{0,3}^{(2)}. \tag{101c}$$

3. *Translating Spheres in Asymmetric Motion*

Next, we study the asymmetric motion, that is, the velocity $\boldsymbol{U}^{(1)}$ is in y-direction as

$$\boldsymbol{U}^{(1)} = (0, U^{(1)}, 0)^t. \tag{102}$$

Note that, for $\boldsymbol{r} = (0, 0, r)$, we have

$$\widehat{\mathsf{A}}_{\alpha\beta}\cdot\boldsymbol{U}^{(\beta)} = \begin{bmatrix} Y_{\alpha\beta}^A U_x^{(\beta)} \\ Y_{\alpha\beta}^A U_y^{(\beta)} \\ X_{\alpha\beta}^A U_z^{(\beta)} \end{bmatrix}, \tag{103}$$

from Eq. (6a).

From Faxén's law for the force in Eq. (85) with the disturbance field by Eq. (88) with Eq. (102), we have the force on the particle 2 due to the translating particle 1 as

$$F_i^{(2)} = 6\pi\mu a_2 \Gamma_{2,3}^{(2)} U_i^{(2)} - 6\pi\mu a_2 \left(\frac{3a_1}{4r}\Gamma_{2,3}^{(2)}\Gamma_{2,3}^{(1)} + \frac{1}{4}\frac{a_1^3}{r^3}\Gamma_{2,3}^{(2)}\Gamma_{0,3}^{(1)} + \frac{a_2^2}{4}\frac{a_1}{r^3}\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)} \right) U^{(1)}\delta_{iy}. \tag{104}$$

In terms of the scalar functions $Y_{\alpha\beta}^A$, the force is expressed by

$$F_i^{(2)} = 6\pi\mu a_2 Y_{22}^A(s,\lambda) U^{(2)}\delta_{iy} + 3\pi\mu(a_2 + a_1) Y_{21}^A(s,\lambda) U^{(1)}\delta_{iy}. \tag{105}$$

Therefore,

$$Y_{22}^A(s,\lambda) = \Gamma_{2,3}^{(2)}, \tag{106a}$$
$$Y_{21}^A(s,\lambda) = -\frac{2\lambda}{1+\lambda}\left(\frac{3}{2}\frac{1}{(1+\lambda)s}\Gamma_{2,3}^{(2)}\Gamma_{2,3}^{(1)} + \frac{2}{(1+\lambda)^3 s^3}\left(\Gamma_{2,3}^{(2)}\Gamma_{0,3}^{(1)} + \lambda^2\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)} \right) \right). \tag{106b}$$

From the symmetry of $Y_{\alpha\beta}^A$ in Eq. (7b), we have

$$Y_{11}^A(s,\lambda) = \Gamma_{2,3}^{(1)}, \tag{107a}$$
$$Y_{12}^A(s,\lambda) = -\frac{2}{1+\lambda}\left(\frac{3}{2}\frac{\lambda}{(1+\lambda)s}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)} + \frac{2}{(1+\lambda)^3 s^3}\left(\lambda^3\Gamma_{2,3}^{(1)}\Gamma_{0,3}^{(2)} + \lambda\Gamma_{0,3}^{(1)}\Gamma_{2,3}^{(2)} \right) \right). \tag{107b}$$

From the expression of Y_{12}^A by the coefficients f_k^{YA} in Eq. (71b), we have

$$f_0^{YA} = \Gamma_{2,3}^{(1)}, \tag{108a}$$
$$f_1^{YA} = \frac{3}{2}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\lambda, \tag{108b}$$
$$f_3^{YA} = 2\Gamma_{2,3}^{(1)}\Gamma_{0,3}^{(2)}\lambda^3 + 2\Gamma_{0,3}^{(1)}\Gamma_{2,3}^{(2)}\lambda. \tag{108c}$$

From Faxén's law for the torque in Eq. (86), we have the torque on the particle 2 due to the translating particle 1 as

$$T_i^{(2)} = -6\pi\mu a_2^3 \frac{a_1}{r^2}\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)} U^{(1)}\delta_{ix}. \tag{109}$$

In terms of the scalar functions $Y_{\alpha\beta}^B$, the torque is expressed by

$$T_i^{(2)} = 4\pi\mu a_2^2 Y_{22}^B \delta_{ix} U^{(2)} + \pi\mu(a_2 + a_1)^2 Y_{21}^B \delta_{ix} U^{(1)}. \tag{110}$$

Therefore,

$$Y_{22}^B = 0, \tag{111a}$$
$$Y_{21}^B = \frac{-4}{(1+\lambda)^2}\frac{6\lambda^3}{(1+\lambda)^2 s^2}\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)}. \tag{111b}$$

From the symmetry of $Y_{\alpha\beta}^B$ in Eq. (7c), we have

$$Y_{11}^B = 0, \tag{112a}$$
$$Y_{12}^B = \frac{-4}{(1+\lambda)^2}\frac{-6\lambda}{(1+\lambda)^2 s^2}\Gamma_{0,3}^{(1)}\Gamma_{2,3}^{(2)}. \tag{112b}$$

From the expression of Y^B_{11} and Y^B_{12} by the coefficients f^{YB}_k in Eqs. (72a) and (72b), we have

$$f^{YB}_0 = 0, \quad (113a)$$
$$f^{YB}_1 = 0, \quad (113b)$$
$$f^{YB}_2 = -6\lambda\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,3}. \quad (113c)$$

From Faxén's law for the stresslet in Eq. (87), the stresslet is given by

$$S^{(2)}_{ij} = \frac{20}{3}\pi\mu a_2^3\Gamma^{(2)}_{2,5}\left(\frac{3a_1^3}{4r^4}\Gamma^{(1)}_{0,3} + \Gamma^{(2)}_{0,2}\frac{a_2}{10}\frac{9a_1}{2r^4}\Gamma^{(1)}_{2,3}\right) \times U^{(1)}\left(\delta_{iy}\delta_{jz} + \delta_{jy}\delta_{iz}\right). \quad (114)$$

Note that

$$S^{(2)}_{ij} = 4\pi\mu a_2^2 G^{22}_{ijk}U^{(2)}_k + \pi\mu(a_2+a_1)^2 G^{21}_{ijk}U^{(1)}_k, \quad (115)$$

and

$$G^{(\alpha\beta)}_{ijk}U_k = Y^G_{\alpha\beta}\left(\delta_{iz}\delta_{jy} + \delta_{jz}\delta_{iy}\right)U, \quad (116)$$

for $\boldsymbol{e} = (0,0,1)$ and $\boldsymbol{U} = (0,U,0)$, we have

$$Y^G_{22} = 0, \quad (117a)$$
$$Y^G_{21} = \frac{20\lambda^2}{(1+\lambda)^2}\left(\frac{4\lambda}{(1+\lambda)^4 s^4}\Gamma^{(2)}_{2,5}\Gamma^{(1)}_{0,3} + \frac{12}{5}\frac{\lambda^3}{(1+\lambda)^4 s^4}\Gamma^{(2)}_{0,5}\Gamma^{(1)}_{2,3}\right). \quad (117b)$$

From the symmetry of $Y^G_{\alpha\beta}$ in Eq. (7g), we have

$$Y^G_{11} = 0, \quad (118a)$$
$$Y^G_{12} = \frac{-4}{(1+\lambda)^2}\left(\frac{20\lambda^3}{(1+\lambda)^4 s^4}\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,3} + \frac{12\lambda}{(1+\lambda)^4 s^4}\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,3}\right). \quad (118b)$$

From the expression of Y^G_{12} by the coefficients f^{YG}_k in Eq. (76b), we have

$$f^{YG}_0 = 0, \quad (119a)$$
$$f^{YG}_2 = 0, \quad (119b)$$
$$f^{YG}_4 = 20\lambda^3\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,3} + 12\lambda\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,3}. \quad (119c)$$

4. $\boldsymbol{\Omega}$ *Problem*

Consider the influence of the particle 1 rotating with the angular velocity $\boldsymbol{\Omega}^{(1)}$. In the two-body problem with $\boldsymbol{r} = (0,0,r)$, we set the angular velocity $\boldsymbol{\Omega}^{(1)}$ for the axisymmetric case by

$$\Omega^{(1)}_i = \Omega^{(1)}\delta_{iz}, \quad (120a)$$

and for the asymmetric case by

$$\Omega^{(1)}_i = \Omega^{(1)}\delta_{iy}. \quad (120b)$$

a. Torque in Axisymmetric Motion From Faxén's law for the torque in Eq. (86) with the disturbance field by Eq. (88) with Eq. (120a), we have the torque on the particle 2 due to the translating particle 1 as

$$T^{(2)}_i = 8\pi\mu a_2^3\Gamma^{(2)}_{0,3}\Omega^{(2)}_i - 8\pi\mu a_2^3\Gamma^{(2)}_{0,3}\frac{a_1^3}{r^3}\Gamma^{(1)}_{0,3}\delta_{iz}\Omega^{(1)}. \quad (121)$$

In terms of the scalar functions $X^C_{\alpha\beta}$, the torque is expressed by

$$T^{(2)}_i = 8\pi\mu a_2^3 X^C_{22}\Omega^{(2)}\delta_{iz} + \pi\mu(a_2+a_1)^3 X^C_{21}\Omega^{(1)}\delta_{iz}. \quad (122)$$

Therefore,

$$X^C_{22} = \Gamma^{(2)}_{0,3}, \quad (123a)$$
$$X^C_{21} = -\frac{8\lambda^3}{(1+\lambda)^3}\frac{8}{(1+\lambda)^3 s^3}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{0,3}. \quad (123b)$$

From the symmetry of $X^C_{\alpha\beta}$ in Eq. (7d), we have

$$X^C_{12}(\lambda) = -\frac{8}{(1+\lambda)^3}\frac{8\lambda^3}{(1+\lambda)^3 s^3}\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}. \quad (124)$$

From the expression of X^C_{12} by the coefficients f^{XC}_k in Eq. (73b), we have

$$f^{XC}_0 = \Gamma^{(1)}_{0,3}, \quad (125a)$$
$$f^{XC}_1 = 0, \quad (125b)$$
$$f^{XC}_3 = 8\lambda^3\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}. \quad (125c)$$

b. Torque in Asymmetric Motion For the asymmetric motion, from Faxén's law for the torque in Eq. (86) with the disturbance field by Eq. (88) with Eq. (120b), we have the torque on the particle 2 due to the translating particle 1 as

$$T^{(2)}_i = 8\pi\mu a_2^3\Gamma^{(2)}_{0,3}\Omega^{(2)}_i + 4\pi\mu a_2^3\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{0,3}\frac{a_1^3}{r^3}\delta_{iy}\Omega^{(1)}. \quad (126)$$

In terms of the scalar functions $Y^C_{\alpha\beta}$, the torque is expressed by

$$T^{(2)}_i = 8\pi\mu a_2^3 Y^C_{22}\Omega^{(2)}\delta_{iy} + \pi\mu(a_2+a_1)^3 Y^C_{21}\Omega^{(1)}\delta_{iy}. \quad (127)$$

Therefore,

$$Y^C_{22} = \Gamma^{(2)}_{0,3}, \quad (128a)$$
$$Y^C_{21} = \frac{4\lambda^3}{(1+\lambda)^3}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{0,3}\frac{8}{(1+\lambda)^3 s^3}. \quad (128b)$$

From the symmetry of $Y^C_{\alpha\beta}$ in Eq. (7e), we have

$$Y^C_{12}(\lambda) = \frac{4}{(1+\lambda)^3}\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}\frac{8\lambda^3}{(1+\lambda)^3 s^3}. \quad (129)$$

From the expression of Y^C_{12} by the coefficients f^{YC}_k in Eq. (74b), we have

$$f^{YC}_1 = 0, \quad (130a)$$
$$f^{YC}_3 = 4\lambda^3\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}. \quad (130b)$$

c. Stresslet Because $\partial_j u_i^{(1)}$ for the axisymmetric motion is anti-symmetric for i and j, there is no contribution for the stresslet. For the asymmetric motion, from Faxén's law for the stresslet in Eq. (87) with the disturbance field by Eq. (88),

$$S_{ij}^{(2)} = 10\pi\mu a_2^3 \frac{a_1^3}{r^3}\Gamma_{2,5}^{(2)}\Gamma_{0,3}^{(1)}\left(\delta_{iz}\delta_{jx}+\delta_{ix}\delta_{jz}\right)\Omega^{(1)}. \tag{131}$$

The stresslet is given by

$$\pi\mu(a_2+a_1)^3 H_{ijk}^{(21)}\Omega_k^{(1)}, \tag{132}$$

and

$$H_{ijk}^{(21)}\Omega_k^{(1)} = Y_{21}^H\left(\delta_{iz}\delta_{jx}+\delta_{jz}\delta_{ix}\right)\Omega^{(1)}, \tag{133}$$

for $\boldsymbol{r}=(0,0,r)$ and $\Omega_k^{(1)}=\Omega^{(1)}\delta_{ky}$. Therefore,

$$Y_{21}^H = \frac{10}{(1+\lambda)^3}\frac{8}{(1+\lambda)^3 s^3}\Gamma_{2,5}^{(2)}\Gamma_{0,3}^{(1)}. \tag{134}$$

From the symmetry of $Y_{\alpha\beta}^H$ in Eq. (7h), we have

$$Y_{12}^H = \frac{10}{(1+\lambda)^3}\frac{8\lambda^3}{(1+\lambda)^3 s^3}\Gamma_{2,5}^{(1)}\Gamma_{0,3}^{(2)}. \tag{135}$$

From the expression of Y_{12}^H by the coefficients f_k^{YH} in Eq. (77b), we have

$$f_1^{YH} = 0, \tag{136a}$$
$$f_3^{YH} = 10\lambda^3\Gamma_{2,5}^{(1)}\Gamma_{0,3}^{(2)}. \tag{136b}$$

5. $\boldsymbol{E}$ Problem

From Faxén's law for the stresslet in Eq. (87),

$$S_{ij}^{(2)} = \frac{20}{3}\pi\mu a_2^3\Gamma_{2,5}^{(2)}\left[E_{ij}^{(2)} - \left(1+\Gamma_{0,2}^{(2)}\frac{a_2^2\nabla^2}{10}\right)\frac{1}{2}\left[\partial_i u_j^{(1)}+\partial_j u_i^{(1)}\right](\boldsymbol{x}_2)\right]. \tag{137}$$

We will see the second term which relates to the resistance functions X_{12}^M, Y_{12}^M, and Z_{12}^M. Let us define three types of strain by

$$E_{kl}^X = E^X\left(\delta_{kz}\delta_{lz}-\frac{\delta_{kl}}{3}\right), \tag{138a}$$
$$E_{kl}^Y = E^Y\left(\delta_{kz}\delta_{lx}+\delta_{kx}\delta_{lz}\right), \tag{138b}$$
$$E_{kl}^Z = E^Z\left(\delta_{kx}\delta_{lx}-\delta_{ky}\delta_{ly}\right), \tag{138c}$$

which correspond to the scalar functions $X_{\alpha\beta}^M$, $Y_{\alpha\beta}^M$, and $Z_{\alpha\beta}^M$, respectively.

a. Function X^M Substituting the disturbance field in Eq. (88) with E_{kl}^X in Eq. (138a) into the second term in Eq. (137), we have

$$\frac{20}{3}\pi\mu a_2^3\left[\frac{5a_1^3}{r^3}\Gamma_{2,5}^{(2)}\Gamma_{2,5}^{(1)} - \frac{6}{r^5}\left(a_1^5\Gamma_{2,5}^{(2)}\Gamma_{0,5}^{(1)}+a_2^2a_1^3\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right)\right] \times\left(\delta_{iz}\delta_{jz}-\frac{\delta_{ij}}{3}\right)E^X. \tag{139}$$

In terms of the scalar function X_{21}^M, it is written by

$$\frac{5}{6}\pi\mu(a_2+a_1)^3 X_{21}^M\left(\delta_{iz}\delta_{jz}-\frac{\delta_{ij}}{3}\right)E^X. \tag{140}$$

Therefore,

$$X_{21}^M = 8\frac{\lambda^3}{(1+\lambda)^3}\left[\frac{40}{(1+\lambda)^3 s^3}\Gamma_{2,5}^{(2)}\Gamma_{2,5}^{(1)} - \frac{192}{(1+\lambda)^5 s^5}\left(\Gamma_{2,5}^{(2)}\Gamma_{0,5}^{(1)}+\lambda^2\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right)\right]. \tag{141}$$

From the symmetry of $X_{\alpha\beta}^M$ in Eq. (7i), we have

$$X_{12}^M = \frac{8}{(1+\lambda)^3}\left[\frac{40\lambda^3}{(1+\lambda)^3 s^3}\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)} - \frac{192}{(1+\lambda)^5 s^5}\left(\lambda^5\Gamma_{2,5}^{(1)}\Gamma_{0,5}^{(2)}+\lambda^3\Gamma_{0,5}^{(1)}\Gamma_{2,5}^{(2)}\right)\right]. \tag{142}$$

From the expression of X_{12}^M by the coefficients f_k^{XM} in Eq. (78b), we have

$$f_1^{XM} = 0, \tag{143a}$$
$$f_3^{XM} = 40\lambda^3\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)}, \tag{143b}$$
$$f_5^{XM} = -192\left(\lambda^5\Gamma_{2,5}^{(1)}\Gamma_{0,5}^{(2)}+\lambda^3\Gamma_{0,5}^{(1)}\Gamma_{2,5}^{(2)}\right). \tag{143c}$$

b. Function Y^M Substituting the disturbance field in Eq. (88) with E_{kl}^Y in Eq. (138b) into the second term in Eq. (137), we have

$$\frac{20}{3}\pi\mu a_2^3\left[-\frac{5}{2}\frac{a_1^3}{r^3}\Gamma_{2,5}^{(2)}\Gamma_{2,5}^{(1)} + \frac{4}{r^5}\left(a_1^5\Gamma_{2,5}^{(2)}\Gamma_{0,5}^{(1)}+a_2^2a_1^3\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right)\right] \times\left(\delta_{ix}\delta_{jz}+\delta_{iz}\delta_{jx}\right)E^Y. \tag{144}$$

In terms of the scalar function Y_{21}^M, it is written by

$$\frac{5}{6}\pi\mu(a_2+a_1)^3 Y_{21}^M\left(\delta_{iz}\delta_{jx}+\delta_{ix}\delta_{jz}\right)E^Y. \tag{145}$$

Therefore,

$$Y_{21}^M = 8\frac{\lambda^3}{(1+\lambda)^3}\left[-\frac{20}{(1+\lambda)^3 s^3}\Gamma_{2,5}^{(2)}\Gamma_{2,5}^{(1)} + \frac{128}{(1+\lambda)^5 s^5}\left(\Gamma_{2,5}^{(2)}\Gamma_{0,5}^{(1)}+\lambda^2\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right)\right]. \tag{146}$$

From the symmetry of $Y^M_{\alpha\beta}$ in Eq. (7j), we have

$$Y^M_{12} = \frac{8}{(1+\lambda)^3}\left[-\frac{20\lambda^3}{(1+\lambda)^3 s^3}\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{2,5} + \frac{128}{(1+\lambda)^5 s^5}\left(\lambda^5\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,5} + \lambda^3\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,5}\right)\right]. \quad (147)$$

From the expression of Y^M_{12} by the coefficients f^{YM}_k in Eq. (79b), we have

$$f^{YM}_1 = 0, \quad (148a)$$
$$f^{YM}_3 = -20\lambda^3\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{2,5}, \quad (148b)$$
$$f^{YM}_5 = 128\left(\lambda^5\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,5} + \lambda^3\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,5}\right). \quad (148c)$$

c. Function Z^M Substituting the disturbance field in Eq. (88) with E^Z_{kl} in Eq. (138c) into the second term in Eq. (137), we have

$$-\frac{20}{3}\pi\mu a_2^3\frac{1}{r^5}\left(a_1^5\Gamma^{(2)}_{2,5}\Gamma^{(1)}_{0,5} + a_2^2a_1^3\Gamma^{(2)}_{0,5}\Gamma^{(1)}_{2,5}\right)\left(\delta_{ix}\delta_{jx} - \delta_{iy}\delta_{jy}\right)E^Z. \quad (149)$$

In terms of the scalar function Z^M_{21}, it is written by

$$\frac{5}{6}\pi\mu(a_2+a_1)^3 Z^M_{21}\left(\delta_{ix}\delta_{jx} - \delta_{iy}\delta_{jy}\right)E^Z. \quad (150)$$

Therefore,

$$Z^M_{21} = -8\frac{\lambda^3}{(1+\lambda)^3}\frac{32}{(1+\lambda)^5 s^5}\left(\Gamma^{(2)}_{2,5}\Gamma^{(1)}_{0,5} + \lambda^2\Gamma^{(2)}_{0,5}\Gamma^{(1)}_{2,5}\right). \quad (151)$$

From the symmetry of $Z^M_{\alpha\beta}$ in Eq. (7k), we have

$$Z^M_{12} = \frac{-8}{(1+\lambda)^3}\frac{32}{(1+\lambda)^5 s^5}\left(\lambda^5\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,5} + \lambda^3\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,5}\right). \quad (152)$$

From the expression of Z^M_{12} by the coefficients f^{ZM}_k in Eq. (80b), we have

$$f^{ZM}_1 = 0, \quad (153a)$$
$$f^{ZM}_3 = 0, \quad (153b)$$
$$f^{ZM}_5 = 32\left(\lambda^5\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{0,5} + \lambda^3\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,5}\right). \quad (153c)$$

B. Twin Multipole Expansions

First, we outline the derivation of equations among coefficients (p_{mn}, q_{mn}, v_{mn}) and $(\psi_{mn}, \chi_{mn}, \omega_{mn})$ for the slip spheres.

1. Outline

In Sec. III, the problem of single slip sphere has been solved by Lamb's general solution in Eq. (10) through three scalars of the surface vector on the both sides of the slip boundary condition in Eq. (16). Jeffrey *et al.*[13,14] solved two-sphere problem with no-slip boundary condition, *i.e.*, (16) with $\gamma = 0$. To complete the boundary condition for two slip spheres, we need to obtain the tangential force density caused by particle $(3-\alpha)$ on the surface of particle α. Let us denote it by $\boldsymbol{t}'^{(\alpha)}$ as

$$\boldsymbol{t}'^{(\alpha)} := \left(\boldsymbol{I} - \boldsymbol{n}^{(\alpha)}\boldsymbol{n}^{(\alpha)}\right)\cdot\left(\boldsymbol{\sigma}^{(3-\alpha)}\cdot\boldsymbol{n}^{(\alpha)}\right), \quad (154)$$

where $\boldsymbol{n}^{(\alpha)}$ is the surface normal of particle α ($\boldsymbol{r}^{(\alpha)}/r_\alpha$ for sphere) and $\boldsymbol{\sigma}^{(3-\alpha)}$ is the disturbance part of the stress caused by particle $(3-\alpha)$ given by

$$\boldsymbol{\sigma}^{(3-\alpha)} = -p^{(3-\alpha)}\boldsymbol{I} + \mu\left[\boldsymbol{\nabla}\boldsymbol{v}^{(3-\alpha)} + \left(\boldsymbol{\nabla}\boldsymbol{v}^{(3-\alpha)}\right)^\dagger\right]. \quad (155)$$

Here, $p^{(3-\alpha)}$ and $\boldsymbol{v}^{(3-\alpha)}$ are expressed by Lamb's solution in Eqs. (9) and (10), respectively. Because $\boldsymbol{\sigma}^{(3-\alpha)}\cdot\boldsymbol{n}^{(\alpha)} \neq \boldsymbol{f}^{(3-\alpha)}$, we cannot use the expression in Eq. (21).

Following similar calculations by Jeffrey and Onishi[13] for the disturbance velocity $\boldsymbol{v}$ caused by $(3-\alpha)$ particle on α particle, $\boldsymbol{\sigma}^{(3-\alpha)}$ can be expressed by the spherical harmonics with respect to the particle α by the transformation [in (JO-2.1)]

$$\left(\frac{a_\alpha}{r_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha,\phi) = \left(\frac{a_\alpha}{r}\right)^{n+1}\sum_{s=m}^{\infty}\begin{pmatrix} n+s \\ s+m\end{pmatrix}\left(\frac{r_{3-\alpha}}{r}\right)^s Y_{ms}(\theta_{3-\alpha},\phi), \quad (156)$$

and the following relations [in (JO-2.7)]

$$\boldsymbol{r}_\alpha = \hat{\boldsymbol{r}}_{3-\alpha}(r_{3-\alpha} - r\cos\theta_{3-\alpha}) + \hat{\boldsymbol{\theta}}_{3-\alpha} r\sin\theta_{3-\alpha} \quad (157a)$$

$$r_\alpha^2 = r_{3-\alpha}^2 + r^2 - 2r_{3-\alpha}r\cos\theta_{3-\alpha}. \quad (157b)$$

After substituting the expansions for the solid spherical harmonics $p^{(3-\alpha)}_{-n-1}$, $\Phi^{(3-\alpha)}_{-n-1}$, and $\chi^{(3-\alpha)}_{-n-1}$ in Eqs. (11a), (11b), and (11c), the three scalars of the surface vector of $\boldsymbol{t}'^{(\alpha)}$ are obtained by the summations for $Y_{mn}(\theta_\alpha,\phi)$. Combining the results for $\boldsymbol{t}'^{(\alpha)}$ with those for the single sphere in Eqs. (43a), (43b), and (43c) and the corresponding results for $\boldsymbol{v}$ given by Jeffrey and Onishi,[13] we have three equations for the coefficients, corresponding to Eqs. (JO-2.9a), (JO-2.9b), and (JO-2.9c) for the no-slip case, as

$$\begin{aligned}&\psi^{(\alpha)}_{mn} - (n-1)\left(1 - 2(n+1)\widehat{\gamma}_\alpha\right)\chi^{(\alpha)}_{mn} \\ &= (n+1)(2n+1)\left(1+2\widehat{\gamma}_\alpha\right)v^{(\alpha)}_{mn} - \frac{n+1}{2}p^{(\alpha)}_{mn} \\ &\quad + \sum_{s=m}^{\infty}\begin{pmatrix} n+s \\ n+m\end{pmatrix} t_\alpha^{n-1}t_{3-\alpha}^s\frac{n}{2n+3}\left(1-(2n+1)\widehat{\gamma}_\alpha\right)p^{(3-\alpha)}_{ms}t_\alpha^2,\end{aligned} \quad (158a)$$

$$\begin{aligned}&\psi^{(\alpha)}_{mn} + (n+2)\left(1+2n\widehat{\gamma}_\alpha\right)\chi^{(\alpha)}_{mn} \\ &= \frac{n+1}{2n-1}\left(1+(2n+1)\widehat{\gamma}_\alpha\right)p^{(\alpha)}_{mn} + \sum_{s=m}^{\infty}\begin{pmatrix} n+s \\ n+m\end{pmatrix} t_\alpha^{n-1}t_{3-\alpha}^s \\ &\quad\times\left[\mathrm{i}(-1)^\alpha m(2n+1)\left(1+2\widehat{\gamma}_\alpha\right)q^{(3-\alpha)}_{ms}t_{3-\alpha}\right. \\ &\quad + n(2n+1)\left(1+2\widehat{\gamma}_\alpha\right)v^{(3-\alpha)}_{ms}t_{3-\alpha}^2 \\ &\quad + \frac{2n+1}{2n-1}\left(1+2\widehat{\gamma}_\alpha\right) \\ &\quad \times\frac{ns(n+s-2ns-2) - m^2(2ns-4s-4n+2)}{2s(2s-1)(n+s)}p^{(3-\alpha)}_{ms} \\ &\quad \left. + \frac{n}{2}p^{(3-\alpha)}_{ms}t_\alpha^2\right],\end{aligned} \quad (158b)$$

$$\omega_{mn}^{(\alpha)} = n(n+1)\left(1+(n+2)\widehat{\gamma}_\alpha\right) q_{mn}^{(\alpha)} + \sum_{s=m}^{\infty} \begin{pmatrix} n+s \\ n+m \end{pmatrix} t_\alpha^n t_{3-\alpha}^s \left(1-(n-1)\widehat{\gamma}_\alpha\right) \times \left[-nsq_{ms}^{(3-\alpha)} t_{3-\alpha} + \mathrm{i}(-1)^\alpha \frac{m}{s} p_{ms}^{(3-\alpha)}\right], \quad (158c)$$

where

$$t_\alpha := \frac{a_\alpha}{r}. \quad (159)$$

Here, we extend (JO-2.9a) and (JO-2.9b) for the slip particles keeping the properties that all terms except for $p_{mn}^{(3-\alpha)}$ in the summation are vanished in Eq. (158a) and $v_{mn}^{(\alpha)}$ is eliminated in Eq. (158b).

Note that Keh and Chen[16] take a different form for the first equation, that is, $\psi_{mn}^{(\alpha)} - \left((n-1)+(2n^2+1)\widehat{\gamma}_\alpha\right)\chi_{mn}^{(\alpha)}$ in Eq. (KC-20a). Although they are mathematically equivalent, Eq. (158a) is simpler and we will use it later in this paper. Also note that there are typos in Keh and Chen[16] at Eqs. (KC-20a,b,c) where $\widehat{\beta}_{(3-\alpha)}^{-1}$ ($\widehat{\gamma}_{(3-\alpha)}$ in the present notation) should be replaced by $\widehat{\beta}_{(\alpha)}^{-1}$. If we look at the slip boundary condition from which these three equations are derived, it is obvious that only the slip length of particle α would appear there. It should be noted that the results such as coefficients f_k in Keh and Chen[16] are correct, because they took a simplification that the scaled slip lengths for two particles are the same as $\widehat{\gamma}_1 = \widehat{\gamma}_2$ in the present notation.

2. Recurrence Relations

For resistance functions, the boundary conditions are given completely by χ_{mn}, ψ_{mn}, and ω_{mn}, which are independent of the distance between the particle r and therefore t_α and $t_{3-\alpha}$. This means that the coefficients P_{npq}, V_{npq}, and Q_{npq} of the (p,q)-term in the expansion by $t_\alpha^p t_{3-\alpha}^q$ (see, for example, Eqs. (163a) and (163b) in the following) are solved by the recurrence relations for $p \geq 0$ and $q \geq 0$ with the initial condition for $p = 0$ and $q = 0$. Therefore, we split the above three equations into two parts, the initial conditions and the recurrence relations. The initial conditions are

$$p_{mn}^{(\alpha)} = \frac{2n-1}{n+1}\Gamma_{0,2n+1}^{(\alpha)}\psi_{mn}^{(\alpha)} + \frac{(n+2)(2n-1)}{n+1}\Gamma_{2n,2n+1}^{(\alpha)}\chi_{mn}^{(\alpha)}, \quad (160a)$$

$$2(2n+1)v_{mn}^{(\alpha)} = \frac{2}{n+1}\Gamma_{0,2}^{(\alpha)}\psi_{mn}^{(\alpha)} - \frac{2(n-1)}{(n+1)}\Gamma_{-2(n+1),2}^{(\alpha)}\chi_{mn}^{(\alpha)} + \Gamma_{0,2}^{(\alpha)} p_{mn}^{(\alpha)}, \quad (160b)$$

$$q_{mn}^{(\alpha)} = \frac{1}{n(n+1)}\Gamma_{0,n+2}^{(\alpha)}\omega_{mn}^{(\alpha)}. \quad (160c)$$

The recurrence relations are

$$p_{mn}^{(\alpha)} = \sum_{s=m}^{\infty} \begin{pmatrix} n+s \\ n+m \end{pmatrix} \times \left[-\mathrm{i}(-1)^\alpha m \frac{(2n+1)(2n-1)}{n+1}\Gamma_{2,2n+1}^{(\alpha)} q_{ms}^{(3-\alpha)} t_\alpha^{n-1} t_{3-\alpha}^{s+1} - \frac{n(2n+1)(2n-1)}{n+1}\Gamma_{2,2n+1}^{(\alpha)} v_{ms}^{(3-\alpha)} t_\alpha^{n-1} t_{3-\alpha}^{s+2} - \frac{2n+1}{n+1}\frac{ns(n+s-2ns-2)-m^2(2ns-4s-4n+2)}{2s(2s-1)(n+s)} \times\Gamma_{2,2n+1}^{(\alpha)} p_{ms}^{(3-\alpha)} t_\alpha^{n-1} t_{3-\alpha}^{s} - \frac{n(2n-1)}{2(n+1)}\Gamma_{0,2n+1}^{(\alpha)} p_{ms}^{(3-\alpha)} t_\alpha^{n+1} t_{3-\alpha}^{s}\right], \quad (161a)$$

$$2(2n+1)v_{mn}^{(\alpha)} = \Gamma_{0,2}^{(\alpha)} p_{mn}^{(\alpha)} - \sum_{s=m}^{\infty} \begin{pmatrix} n+s \\ n+m \end{pmatrix} \frac{2n}{(n+1)(2n+3)}\Gamma_{-(2n+1),2}^{(\alpha)} p_{ms}^{(3-\alpha)} t_\alpha^{n+1} t_{3-\alpha}^{s}, \quad (161b)$$

$$q_{mn}^{(\alpha)} = \sum_{s=m}^{\infty} \begin{pmatrix} n+s \\ n+m \end{pmatrix} \left[\frac{s}{(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} q_{ms}^{(3-\alpha)} t_\alpha^{n} t_{3-\alpha}^{s+1} - \mathrm{i}(-1)^\alpha \frac{m}{ns(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} p_{ms}^{(3-\alpha)} t_\alpha^{n} t_{3-\alpha}^{s}\right]. \quad (161c)$$

It should be noted that the initial conditions are independent of m, while the recurrence relations are not. Therefore, the initial conditions are the same for X ($m = 0$), Y ($m = 1$), and Z ($m = 2$) functions for each problems (translating, rotating, or in the shear).

Note that the recurrence relations have α-dependent quantity $\Gamma^{(\alpha)}$, so that we need to solve the coefficients P_{npq}, V_{npq}, and Q_{npq} for α as well as $3-\alpha$, while, for the no-slip case, the coefficients for α and $3-\alpha$ are identical.

The results shown in the following are obtained by the program implemented on an open source computer algebra system called "Maxima" (`http://maxima.sourceforge.net/`), so that the results are exact. The program is relatively slow due to its symbolic calculation and the coefficients are obtained up to $k = 20$, at least. We also implement a code in C with floating-point variables where the parameters a_α and γ_α must be given by numbers for the calculation. With this code, we can obtain the coefficients around $k = 100$.

C. X Functions ($m = 0$)

For the case of $m = 0$, $q^{(\alpha)}$ and $q^{(3-\alpha)}$ are decoupled with others.

1. X^A Function

The boundary condition for the X^A problem is given by

$$\chi_{mn}^{(\alpha)} = U\delta_{m0}\delta_{n1}, \quad \psi_{mn}^{(\alpha)} = 0, \quad \omega_{mn}^{(\alpha)} = 0. \quad (162)$$

To obtain the coefficients for each order of the power of r, we expand the coefficients [in (JO-3.4) and (JO-3.5)] as

$$p_{0n}^{(\alpha)} = \frac{3}{2}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}P_{npq}^{(\alpha)}t_{\alpha}^{p}t_{3-\alpha}^{q}, \qquad (163a)$$

$$v_{0n}^{(\alpha)} = \frac{3}{2}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{V_{npq}^{(\alpha)}}{2(2n+1)}t_{\alpha}^{p}t_{3-\alpha}^{q}. \qquad (163b)$$

Substituting the expansions, we have the initial conditions for $p = 0$ and $q = 0$ from Eqs. (160a) and (160b) by

$$P_{n00}^{(\alpha)} = \delta_{n1}\Gamma_{2,3}^{(\alpha)}, \quad V_{n00}^{(\alpha)} = \delta_{n1}\Gamma_{0,3}^{(\alpha)}, \qquad (164)$$

and the recurrence relations for $p \geq 0$ and $q \geq 0$ from Eqs. (161a) and (161b) by

$$\begin{aligned} P_{npq}^{(\alpha)} = \sum_{s=0}^{\infty}\binom{n+s}{n} & \times\Bigg[-\frac{n(2n-1)(2n+1)}{2(n+1)(2s+1)}\Gamma_{2,2n+1}^{(\alpha)}V_{s(q-s-2)(p-n+1)}^{(3-\alpha)} \\ & -\frac{n(2n+1)(n+s-2ns-2)}{2(n+1)(n+s)(2s-1)}\Gamma_{2,2n+1}^{(\alpha)}P_{s(q-s)(p-n+1)}^{(3-\alpha)} \\ & -\frac{n(2n-1)}{2(n+1)}\Gamma_{0,2n+1}^{(\alpha)}P_{s(q-s)(p-n-1)}^{(3-\alpha)}\Bigg]. \end{aligned} \qquad (165a)$$

$$\begin{aligned} V_{npq}^{(\alpha)} &= \Gamma_{0,2}^{(\alpha)}P_{npq}^{(\alpha)} \\ &- \sum_{s=0}^{\infty}\binom{n+s}{n}\frac{2n}{(n+1)(2n+3)}\Gamma_{-(2n+1),2}^{(\alpha)}P_{s(q-s)(p-n-1)}^{(3-\alpha)}. \end{aligned} \qquad (165b)$$

Note that the initial conditions correspond to Eqs. (KC-26a,b) and the recurrence relations except for $V_{npq}^{(\alpha)}$ to Eqs. (KC-27a,c). Equation (163b) is simpler than the corresponding equation in Keh and Chan (KC-27b), because we use the simpler recurrence relation in Eq. (158a).

The coefficient $f_k^{XA\alpha}$ is defined [in (JO-3.15)] by

$$f_k^{XA\alpha} := 2^k\sum_{q=0}^{k}P_{1(k-q)q}^{(\alpha)}\lambda^q. \qquad (166)$$

Here we see a slight difference from the no-slip case. This is because of the α dependence of $P_{npq}^{(\alpha)}$, and the coefficient $f_k^{XA\alpha}$ also depends on α. The explicit forms up to $k = 7$ are

$$f_0^{XA1} = \left(\Gamma_{2,3}^{(1)}\right), \qquad (167a)$$

$$f_1^{XA1} = \lambda\left(3\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\right), \qquad (167b)$$

$$f_2^{XA1} = \lambda\left(9(\Gamma_{2,3}^{(1)})^2\Gamma_{2,3}^{(2)}\right), \qquad (167c)$$

$$\begin{aligned} f_3^{XA1} &= \lambda\left(-4\Gamma_{0,3}^{(1)}\Gamma_{2,3}^{(2)}\right) \\ &+ \lambda^2\left(27(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^2\right) \\ &+ \lambda^3\left(-4\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)}\right), \end{aligned} \qquad (167d)$$

$$\begin{aligned} f_4^{XA1} &= \lambda\left(-24\Gamma_{0,3}^{(1)}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\right) \\ &+ \lambda^2\left(81(\Gamma_{2,3}^{(1)})^3(\Gamma_{2,3}^{(2)})^2\right) \\ &+ \lambda^3\left(12(\Gamma_{2,3}^{(1)})^2(5\Gamma_{2,5}^{(2)} - 2\Gamma_{0,3}^{(2)})\right), \end{aligned} \qquad (167e)$$

$$\begin{aligned} f_5^{XA1} &= \lambda^2\left(36\Gamma_{2,3}^{(1)}(\Gamma_{2,3}^{(2)})^2(5\Gamma_{2,5}^{(1)} - 3\Gamma_{0,3}^{(1)})\right) \\ &+ \lambda^3\left(243(\Gamma_{2,3}^{(1)})^3(\Gamma_{2,3}^{(2)})^3\right) \\ &+ \lambda^4\left(36(\Gamma_{2,3}^{(1)})^2\Gamma_{2,3}^{(2)}(5\Gamma_{2,5}^{(2)} - 3\Gamma_{0,3}^{(2)})\right), \end{aligned} \qquad (167f)$$

$$\begin{aligned} f_6^{XA1} &= \lambda\left(16(\Gamma_{0,3}^{(1)})^2\Gamma_{2,3}^{(2)}\right) \\ &+ \lambda^2\left(108(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^2(5\Gamma_{2,5}^{(1)} - 4\Gamma_{0,3}^{(1)})\right) \\ &+ \lambda^3\left(-\Gamma_{2,3}^{(1)}(480\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(2)} - 729(\Gamma_{2,3}^{(1)})^3(\Gamma_{2,3}^{(2)})^3\right. \\ &\quad \left.-32\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)})\right) \\ &+ \lambda^4\left(216(\Gamma_{2,3}^{(1)})^3\Gamma_{2,3}^{(2)}(5\Gamma_{2,5}^{(2)} - 2\Gamma_{0,3}^{(2)})\right) \\ &+ \lambda^5\left(16(\Gamma_{2,3}^{(1)})^2(126\Gamma_{2,7}^{(2)} - 90\Gamma_{0,5}^{(2)} + 5\Gamma_{0,2}^{(2)}\Gamma_{0,3}^{(2)}\right. \\ &\quad \left.+4\Gamma_{-3,2}^{(2)})/5\right), \end{aligned} \qquad (167g)$$

$$\begin{aligned} f_7^{XA1} &= \lambda^2\left(48(\Gamma_{2,3}^{(2)})^2(126\Gamma_{2,3}^{(1)}\Gamma_{2,7}^{(1)} - 70\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(1)}\right. \\ &\quad \left.-45\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)} + 15(\Gamma_{0,3}^{(1)})^2 + 4\Gamma_{-3,3}^{(1)})/5\right) \\ &+ \lambda^3\left(1620(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^3(2\Gamma_{2,5}^{(1)} - \Gamma_{0,3}^{(1)})\right) \\ &+ \lambda^4\left(3\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}(800\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)} - 560\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(2)}\right. \\ &\quad \left.-560\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(1)} + 729(\Gamma_{2,3}^{(1)})^3(\Gamma_{2,3}^{(2)})^3 + 96\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)})\right) \\ &+ \lambda^5\left(1620(\Gamma_{2,3}^{(1)})^3(\Gamma_{2,3}^{(2)})^2(2\Gamma_{2,5}^{(2)} - \Gamma_{0,3}^{(2)})\right) \\ &+ \lambda^6\left(48(\Gamma_{2,3}^{(1)})^2(126\Gamma_{2,3}^{(2)}\Gamma_{2,7}^{(2)} - 70\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(2)} - 45\Gamma_{0,5}^{(2)}\Gamma_{2,3}^{(2)}\right. \\ &\quad \left.+15(\Gamma_{0,3}^{(2)})^2 + 4\Gamma_{-3,3}^{(2)})/5\right). \end{aligned} \qquad (167h)$$

The results are identical to those obtained by method of reflections in Eqs. (95c), (95a), and (95b) for the terms containing one or two Γ's, because only the first influence from the particles 1 to 2 is taken and the higher reflections are missing in the present calculation of the method of reflections. Therefore, f_2^{XA1} and λ^2 term in f_3^{XA} do not appear.

Also the results reduce to those by Jeffrey and Onishi[13] in the no-slip limit $\widehat{\gamma} = 0$, and those by Keh and Chen[16] in the case of $\widehat{\gamma}_1 = \widehat{\gamma}_2$. Therefore, they also reduce to those by Hetsroni and Haber[21] in the perfect slip limit $\widehat{\gamma} = \infty$.

2. X^G Function

With the same recurrence relations and the initial condition for X^A, that is, for the translating particles, the function X^G is obtained from the coefficient P_{2pq} for the stresslet instead of P_{1pq} for the force.

In this case, the coefficient $f_k^{XG\alpha}$ is defined by

$$f_k^{XG\alpha} := \left(\frac{3}{4}\right) 2^k \sum_{q=0}^{k} P^{(\alpha)}_{2(k-q)q} \lambda^q. \tag{168}$$

The explicit forms up to $k = 7$ are

$$f_0^{XG1} = 0, \tag{169a}$$
$$f_1^{XG1} = 0, \tag{169b}$$
$$f_2^{XG1} = \lambda\left(15\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,5}\right), \tag{169c}$$
$$f_3^{XG1} = \lambda\left(45\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,5}\right), \tag{169d}$$
$$\begin{aligned} f_4^{XG1} &= \lambda\left(-36\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,3}\right) \\ &+ \lambda^2\left(135\Gamma^{(1)}_{2,3}(\Gamma^{(2)}_{2,3})^2\Gamma^{(1)}_{2,5}\right) \\ &+ \lambda^3\left(-60\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,5}\right), \end{aligned} \tag{169e}$$
$$\begin{aligned} f_5^{XG1} &= \lambda\left(-12\Gamma^{(2)}_{2,3}(5\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,5} + 9\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,3})\right) \\ &+ \lambda^2\left(405(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^2\Gamma^{(1)}_{2,5}\right) \\ &+ \lambda^3\left(120\Gamma^{(1)}_{2,3}\Gamma^{(1)}_{2,5}(5\Gamma^{(2)}_{2,5} - 2\Gamma^{(2)}_{0,3})\right), \end{aligned} \tag{169f}$$
$$\begin{aligned} f_6^{XG1} &= \lambda^2\left(36(\Gamma^{(2)}_{2,3})^2(25(\Gamma^{(1)}_{2,5})^2 - 10\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,5} - 9\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,3})\right) \\ &+ \lambda^3\left(1215(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^3\Gamma^{(1)}_{2,5}\right) \\ &+ \lambda^4\left(900\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,5}(2\Gamma^{(2)}_{2,5} - \Gamma^{(2)}_{0,3})\right), \end{aligned} \tag{169g}$$
$$\begin{aligned} f_7^{XG1} &= \lambda\left(144\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,3}\right) \\ &+ \lambda^2\left(108\Gamma^{(1)}_{2,3}(\Gamma^{(2)}_{2,3})^2(25(\Gamma^{(1)}_{2,5})^2 - 15\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,5} - 9\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,3})\right) \\ &+ \lambda^3\left(-3(800\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,5}\Gamma^{(2)}_{2,5} + 960\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,5}\right. \\ &\quad -1215(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^3\Gamma^{(1)}_{2,5} \\ &\quad \left. -80\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,5} - 48\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,3})\right) \\ &+ \lambda^4\left(1620(\Gamma^{(1)}_{2,3})^2\Gamma^{(2)}_{2,3}\Gamma^{(1)}_{2,5}(5\Gamma^{(2)}_{2,5} - 2\Gamma^{(2)}_{0,3})\right) \\ &+ \lambda^5\left(48\Gamma^{(1)}_{2,3}\Gamma^{(1)}_{2,5}(126\Gamma^{(2)}_{2,7} - 90\Gamma^{(2)}_{0,5} + 5\Gamma^{(2)}_{0,2}\Gamma^{(2)}_{0,3}\right. \\ &\quad \left. +4\Gamma^{(2)}_{-3,2})\right). \end{aligned} \tag{169h}$$

The results are identical to those obtained by method of reflections in Eqs. (101a), (101b), and (101c) for the terms containing one or two Γ's as similar to X^A. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

3. X^C Function

The function X^C gives the torque for the rotating particles in the axisymmetric case ($m = 0$). The boundary condition is given by

$$\chi^{(\alpha)}_{mn} = 0, \quad \psi^{(\alpha)}_{mn} = 0, \quad \omega^{(\alpha)}_{mn} = 2U\delta_{m0}\delta_{n1}. \tag{170}$$

Note that Q_{npq} is decoupled with P_{npq} and V_{npq} for $m = 0$. Using the expansion [in (JO-6.4)]

$$q^{(\alpha)}_{0n} = U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q^{(\alpha)}_{npq} t^p_\alpha t^q_{3-\alpha}, \tag{171}$$

we have the initial condition for $p = 0$ and $q = 0$ from Eq. (160c) by

$$Q_{n00} = \delta_{n1}\Gamma^{(\alpha)}_{0,3}, \tag{172}$$

and the recurrence relation for $p \geq 0$ and $q \geq 0$ from Eq. (161c) by

$$Q^{(\alpha)}_{npq} = \sum_{s=0}^{\infty} \binom{n+s}{n} \frac{s}{(n+1)} \Gamma^{(\alpha)}_{-(n-1),n+2} Q^{(3-\alpha)}_{s(q-s-1)(p-n)}. \tag{173}$$

The coefficient $f_k^{XC\alpha}$ is defined by

$$f_k^{XC\alpha} := 2^k \sum_{q=0}^{k} Q^{(\alpha)}_{1(k-q)q} \lambda^{q+j}, \tag{174}$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 11$ are

$$f_0^{XC1} = \left(\Gamma^{(1)}_{0,3}\right), \tag{175a}$$
$$f_1^{XC1} = 0, \tag{175b}$$
$$f_2^{XC1} = 0, \tag{175c}$$
$$f_3^{XC1} = \lambda^3\left(8\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}\right), \tag{175d}$$
$$f_4^{XC1} = 0, \tag{175e}$$
$$f_5^{XC1} = 0, \tag{175f}$$
$$f_6^{XC1} = \lambda^3\left(64(\Gamma^{(1)}_{0,3})^2\Gamma^{(2)}_{0,3}\right), \tag{175g}$$
$$f_7^{XC1} = 0, \tag{175h}$$
$$f_8^{XC1} = \lambda^5\left(768\Gamma^{(2)}_{-1,4}(\Gamma^{(1)}_{0,3})^2\right), \tag{175i}$$
$$f_9^{XC1} = \lambda^6\left(512(\Gamma^{(1)}_{0,3})^2(\Gamma^{(2)}_{0,3})^2\right), \tag{175j}$$
$$f_{10}^{XC1} = \lambda^7\left(6144\Gamma^{(2)}_{-2,5}(\Gamma^{(1)}_{0,3})^2\right), \tag{175k}$$
$$\begin{aligned} f_{11}^{XC1} &= \lambda^6\left(6144\Gamma^{(1)}_{-1,4}\Gamma^{(1)}_{0,3}(\Gamma^{(2)}_{0,3})^2\right) \\ &+ \lambda^8\left(6144\Gamma^{(2)}_{-1,4}(\Gamma^{(1)}_{0,3})^2\Gamma^{(2)}_{0,3}\right). \end{aligned} \tag{175l}$$

The results are identical to those obtained by method of reflections in Eqs. (125a), (125b), and (125c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey and Onishi[13] in the no-slip limit $\widehat{\gamma} = 0$ and those by Keh and Chen[16] in the case of $\widehat{\gamma}_1 = \widehat{\gamma}_2$.

4. X^M Function

The function X^M gives the stresslet under a shear flow in the axisymmetric case ($m = 0$). Therefore, it is derived by the coefficient P_{2pq} for the stresslet from the same recurrence relations for X^A with a different initial condition. The boundary condition is given by

$$\chi_{mn}^{(\alpha)} = \frac{2}{3} a_\alpha E_\alpha \delta_{0m}\delta_{2n}, \tag{176a}$$

$$\psi_{mn}^{(\alpha)} = \frac{2}{3} a_\alpha E_\alpha (1-6\widehat{\gamma})\delta_{0m}\delta_{2n}, \tag{176b}$$

$$\omega_{mn}^{(\alpha)} = 0, \tag{176c}$$

which correspond to Eq. (J-41) with the correction due to the slip. Using the expansion [in (J-42) and (J-43)]

$$p_{0n}^{(\alpha)} = \frac{10}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq}^{(\alpha)} t_\alpha^p t_{3-\alpha}^q, \tag{177a}$$

$$v_{0n}^{(\alpha)} = \frac{10}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{V_{npq}^{(\alpha)}}{2(2n+1)} t_\alpha^p t_{3-\alpha}^q, \tag{177b}$$

the initial conditions for P_{npq} and V_{npq} are given from Eqs. (160a) and (160b) by

$$P_{n00}^{(\alpha)} = \delta_{2n}\Gamma_{2,5}^{(\alpha)}, \quad V_{n00}^{(\alpha)} = \delta_{2n}\Gamma_{0,5}^{(\alpha)}. \tag{178}$$

The coefficient $f_k^{XM\alpha}$ is defined by

$$f_k^{XM\alpha} := 2^k \sum_{q=0}^{k} P_{2(k-q)q}^{(\alpha)} \lambda^{q+j}, \tag{179}$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 7$ are

$$f_0^{XM1} = \left(\Gamma_{2,5}^{(1)}\right), \tag{180a}$$

$$f_1^{XM1} = 0, \tag{180b}$$

$$f_2^{XM1} = 0, \tag{180c}$$

$$f_3^{XM1} = \lambda^3\left(40\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)}\right), \tag{180d}$$

$$f_4^{XM1} = \lambda\left(60\Gamma_{2,3}^{(2)}(\Gamma_{2,5}^{(1)})^2\right), \tag{180e}$$

$$\begin{aligned} f_5^{XM1} &= \lambda^3\left(-192\Gamma_{0,5}^{(1)}\Gamma_{2,5}^{(2)}\right) \\ &+ \lambda^4\left(180\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)}\right) \\ &+ \lambda^5\left(-192\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right), \end{aligned} \tag{180f}$$

$$\begin{aligned} f_6^{XM1} &= \lambda\left(-288\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)}\Gamma_{2,5}^{(1)}\right) \\ &+ \lambda^2\left(540\Gamma_{2,3}^{(1)}(\Gamma_{2,3}^{(2)})^2(\Gamma_{2,5}^{(1)})^2\right) \\ &+ \lambda^3\left(160(\Gamma_{2,5}^{(1)})^2(10\Gamma_{2,5}^{(2)} - 3\Gamma_{0,3}^{(2)})\right), \end{aligned} \tag{180g}$$

$$\begin{aligned} f_7^{XM1} &= \lambda^4\left(48\Gamma_{2,3}^{(2)}(50(\Gamma_{2,5}^{(1)})^2 - 20\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(1)} - 9\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)})\Gamma_{2,5}^{(2)}\right) \\ &+ \lambda^5\left(1620(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^2\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)}\right) \\ &+ \lambda^6\left(48\Gamma_{2,3}^{(1)}\Gamma_{2,5}^{(1)}(50(\Gamma_{2,5}^{(2)})^2 - 20\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(2)} \right.\\ &\left. -9\Gamma_{0,5}^{(2)}\Gamma_{2,3}^{(2)})\right). \end{aligned} \tag{180h}$$

The results are identical to those obtained by method of reflections in Eqs. (143a), (143b), and (143c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

D. Y Functions ($m = 1$)

1. Y^A Functions

The boundary condition for the Y^A problem is given by

$$\chi_{mn}^{(\alpha)} = (-1)^\alpha U\delta_{m1}\delta_{n1}, \quad \psi_{mn}^{(\alpha)} = 0, \quad \omega_{mn}^{(\alpha)} = 0. \tag{181}$$

(Note that the equation in Jeffrey and Onishi[13] in p.271 lost the factor U for $\chi_{mn}^{(\alpha)}$.) Again, we expand the coefficients by t_α^p and $t_{3-\alpha}^q$ as

$$p_{1n}^{(\alpha)} = (-1)^\alpha \frac{3}{2} U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq}^{(\alpha)} t_\alpha^p t_{3-\alpha}^q, \tag{182a}$$

$$v_{1n}^{(\alpha)} = (-1)^\alpha \frac{3}{2} U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{V_{npq}^{(\alpha)}}{2(2n+1)} t_\alpha^p t_{3-\alpha}^q, \tag{182b}$$

$$q_{1n}^{(\alpha)} = -\mathrm{i}U \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{npq}^{(\alpha)} t_\alpha^p t_{3-\alpha}^q. \tag{182c}$$

Also note that in Jeffrey and Onishi,[13] the minus sign in the right-hand side of (JO-4.5) is missing. Substituting these expansions into Eqs. (160a), (160b), and (160c), the initial conditions are given by

$$P_{n00}^{(\alpha)} = \delta_{n1}\Gamma_{2,3}^{(\alpha)}, \quad V_{n00}^{(\alpha)} = \delta_{n1}\Gamma_{0,3}^{(\alpha)}, \quad Q_{n00}^{(\alpha)} = 0, \tag{183}$$

which correspond to (KC-37a,b,c). From Eqs. (161a), (161b), and (161c), the recurrence relations are given by

$$\begin{aligned} P_{npq}^{(\alpha)} &= \sum_{s=1}^{\infty}\binom{n+s}{n+1} \\ &\times\left[-\frac{2}{3}\frac{(2n+1)(2n-1)}{n+1}\Gamma_{2,2n+1}^{(\alpha)} Q_{s(q-s-1)(p-n+1)}^{(3-\alpha)}\right. \\ &+\frac{n(2n+1)(2n-1)}{2(n+1)(2s+1)}\Gamma_{2,2n+1}^{(\alpha)} V_{s(q-s-2)(p-n+1)}^{(3-\alpha)} \\ &+\frac{2n+1}{n+1}\frac{ns(n+s-2ns-2)-(2ns-4s-4n+2)}{2s(2s-1)(n+s)} \\ &\Gamma_{2,2n+1}^{(\alpha)} P_{s(q-s)(p-n+1)}^{(3-\alpha)} \\ &\left.+\frac{n(2n-1)}{2(n+1)}\Gamma_{0,2n+1}^{(\alpha)} P_{s(q-s)(p-n-1)}^{(3-\alpha)}\right]. \end{aligned} \tag{184a}$$

$$\begin{aligned} V_{npq}^{(\alpha)} &= \Gamma_{0,2}^{(\alpha)} P_{npq}^{(\alpha)} \\ &+\sum_{s=1}^{\infty}\binom{n+s}{n+1}\frac{2n}{(n+1)(2n+3)}\Gamma_{-(2n+1),2}^{(\alpha)} P_{s(q-s)(p-n-1)}^{(3-\alpha)}, \end{aligned} \tag{184b}$$

$$\begin{aligned} Q_{npq}^{(\alpha)} &= \sum_{s=1}^{\infty}\binom{n+s}{n+1} \\ &\times\left[\frac{s}{(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} Q_{s(q-s-1)(p-n)}^{(3-\alpha)}\right. \\ &\left.-\frac{3}{2}\frac{1}{ns(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} P_{s(q-s)(p-n)}\right]. \end{aligned} \tag{184c}$$

Note that Eqs. (184a) and (184c) correspond to (KC-38a) and (KC-38b), while Eq. (184b) is simpler than Eq. (KC-38c). The coefficient $f_k^{YA\alpha}$ is defined by

$$f_k^{YA\alpha} := 2^k \sum_{q=0} P^{(\alpha)}_{1(k-q)q} \lambda^q. \tag{185}$$

The explicit forms up to $k = 7$ are

$$\begin{aligned}
f_0^{YA1} &= \left(\Gamma^{(1)}_{2,3}\right), && (186a)\\
f_1^{YA1} &= \lambda\left(3\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}/2\right), && (186b)\\
f_2^{YA1} &= \lambda\left(9(\Gamma^{(1)}_{2,3})^2\Gamma^{(2)}_{2,3}/4\right), && (186c)\\
f_3^{YA1} &= \lambda\left(2\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^2\left(27(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^2/8\right)\\
&+ \lambda^3\left(2\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,3}\right), && (186d)\\
f_4^{YA1} &= \lambda\left(6\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^2\left(81(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^2/16\right)\\
&+ \lambda^3\left(18\Gamma^{(2)}_{0,3}(\Gamma^{(1)}_{2,3})^2\right), && (186e)\\
f_5^{YA1} &= \lambda^2\left(63\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,3}(\Gamma^{(2)}_{2,3})^2/2\right)\\
&+ \lambda^3\left(243(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^3/32\right)\\
&+ \lambda^4\left(63\Gamma^{(2)}_{0,3}(\Gamma^{(1)}_{2,3})^2\Gamma^{(2)}_{2,3}/2\right), && (186f)\\
f_6^{YA1} &= \lambda\left(4(\Gamma^{(1)}_{0,3})^2\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^2\left(54\Gamma^{(1)}_{0,3}(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^2\right)\\
&+ \lambda^3\left(\Gamma^{(1)}_{2,3}(729(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^3 + 512\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3})/64\right)\\
&+ \lambda^4\left(81\Gamma^{(2)}_{0,3}(\Gamma^{(1)}_{2,3})^3\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^5\left(4(\Gamma^{(1)}_{2,3})^2(21\Gamma^{(2)}_{2,7} + 5\Gamma^{(2)}_{0,2}\Gamma^{(2)}_{0,3} + 60\Gamma^{(2)}_{-1,4}\right.\\
&\quad \left. +4\Gamma^{(2)}_{-3,2})/5\right), && (186g)\\
f_7^{YA1} &= \lambda^2\left(6(\Gamma^{(2)}_{2,3})^2(21\Gamma^{(1)}_{2,3}\Gamma^{(1)}_{2,7} + 60\Gamma^{(1)}_{-1,4}\Gamma^{(1)}_{2,3} + 35(\Gamma^{(1)}_{0,3})^2\right.\\
&\quad \left. +4\Gamma^{(1)}_{-3,3})/5\right)\\
&+ \lambda^3\left(1053\Gamma^{(1)}_{0,3}(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^3/8\right)\\
&+ \lambda^4\left(3\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}(729(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^3 + 5632\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3})/128\right)\\
&+ \lambda^5\left(1053\Gamma^{(2)}_{0,3}(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^2/8\right)\\
&+ \lambda^6\left(6(\Gamma^{(1)}_{2,3})^2(21\Gamma^{(2)}_{2,3}\Gamma^{(2)}_{2,7} + 60\Gamma^{(2)}_{-1,4}\Gamma^{(2)}_{2,3} + 35(\Gamma^{(2)}_{0,3})^2\right.\\
&\quad \left. +4\Gamma^{(2)}_{-3,3})/5\right). && (186h)
\end{aligned}$$

The results are identical to those obtained by method of reflections in Eqs. (108a), (108b), and (108c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey and Onishi[13] in the no-slip limit $\widehat{\gamma} = 0$ and those by Keh and Chen[16] in the case of $\widehat{\gamma}_1 = \widehat{\gamma}_2$.

2. Y^B Functions

The problem for Y^B is exactly the same for Y^A. The difference is that the force is calculated in Y^A while the torque in Y^B. Correspondingly, The coefficient $f_k^{YB\alpha}$ is defined by

$$f_k^{YB\alpha} := 2\; 2^k \sum_{q=0} Q^{(\alpha)}_{1(k-q)q} \lambda^q, \tag{187}$$

for Q_{1pq} obtained by the recurrence relations for Y^A. The explicit forms up to $k = 7$ are

$$\begin{aligned}
f_0^{YB1} &= 0, && (188a)\\
f_1^{YB1} &= 0, && (188b)\\
f_2^{YB1} &= \lambda\left(-6\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,3}\right), && (188c)\\
f_3^{YB1} &= \lambda\left(-9\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\right), && (188d)\\
f_4^{YB1} &= \lambda^2\left(-27\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,3}(\Gamma^{(2)}_{2,3})^2/2\right), && (188e)\\
f_5^{YB1} &= \lambda\left(-12(\Gamma^{(1)}_{0,3})^2\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^2\left(-81\Gamma^{(1)}_{0,3}(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^2/4\right)\\
&+ \lambda^3\left(-36\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,3}\right), && (188f)\\
f_6^{YB1} &= \lambda^2\left(-108(\Gamma^{(1)}_{0,3})^2(\Gamma^{(2)}_{2,3})^2\right)\\
&+ \lambda^3\left(-243\Gamma^{(1)}_{0,3}(\Gamma^{(1)}_{2,3})^2(\Gamma^{(2)}_{2,3})^3/8\right)\\
&+ \lambda^4\left(-72\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}\Gamma^{(1)}_{2,3}\Gamma^{(2)}_{2,3}\right), && (188g)\\
f_7^{YB1} &= \lambda^2\left(-189(\Gamma^{(1)}_{0,3})^2\Gamma^{(1)}_{2,3}(\Gamma^{(2)}_{2,3})^2\right)\\
&+ \lambda^3\left(-3\Gamma^{(1)}_{0,3}(2560\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{2,5} + 243(\Gamma^{(1)}_{2,3})^3(\Gamma^{(2)}_{2,3})^3)/16\right)\\
&+ \lambda^4\left(-243\Gamma^{(1)}_{0,3}\Gamma^{(2)}_{0,3}(\Gamma^{(1)}_{2,3})^2\Gamma^{(2)}_{2,3}\right)\\
&+ \lambda^5\left(48\Gamma^{(1)}_{0,3}\Gamma^{(1)}_{2,3}(7\Gamma^{(2)}_{2,7} - 6\Gamma^{(2)}_{0,5} - 4\Gamma^{(2)}_{-1,4})\right). && (188h)
\end{aligned}$$

The results are identical to those obtained by method of reflections in Eqs. (113a), (113b), and (113c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey and Onishi[13] in the no-slip limit $\widehat{\gamma} = 0$ and those by Keh and Chen[16] in the case of $\widehat{\gamma}_1 = \widehat{\gamma}_2$.

3. Y^G Function

With the same recurrence relations and the initial condition for Y^A, that is, for the translating particles, the function Y^G is obtained from the coefficient P_{2pq} for the stresslet instead of P_{1pq} for the force.

In this case, the coefficient $f_k^{YG\alpha}$ is defined by

$$f_k^{YG\alpha} := \left(\frac{3}{4}\right) 2^k \sum_{q=0}^{k} P^{(\alpha)}_{2(k-q)q} \lambda^q. \tag{189}$$

The explicit forms up to $k = 7$ are

$$
\begin{aligned}
f_0^{YG1} &= 0, && \text{(190a)}\\
f_1^{YG1} &= 0, && \text{(190b)}\\
f_2^{YG1} &= 0, && \text{(190c)}\\
f_3^{YG1} &= 0, && \text{(190d)}\\
f_4^{YG1} &= \lambda\left(12\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^3\left(20\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(1)}\right), && \text{(190e)}\\
f_5^{YG1} &= \lambda\left(18\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^3\left(90\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)}\Gamma_{2,5}^{(1)}\right), && \text{(190f)}\\
f_6^{YG1} &= \lambda^2\left(27\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)}(\Gamma_{2,3}^{(2)})^2\right)\\
&+ \lambda^4\left(135\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\Gamma_{2,5}^{(1)}\right), && \text{(190g)}\\
f_7^{YG1} &= \lambda\left(24\Gamma_{0,3}^{(1)}\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^2\left(81\Gamma_{0,5}^{(1)}(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^2/2\right)\\
&+ \lambda^3\left(-8(50\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)} - 5\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(1)} - 3\Gamma_{0,3}^{(2)}\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)})\right)\\
&+ \lambda^4\left(405\Gamma_{0,3}^{(2)}(\Gamma_{2,3}^{(1)})^2\Gamma_{2,3}^{(2)}\Gamma_{2,5}^{(1)}/2\right)\\
&+ \lambda^5\left(8\Gamma_{2,3}^{(1)}\Gamma_{2,5}^{(1)}(56\Gamma_{2,7}^{(2)} - 30\Gamma_{0,5}^{(2)} + 5\Gamma_{0,2}^{(2)}\Gamma_{0,3}^{(2)} + 40\Gamma_{-1,4}^{(2)}\right.\\
&\left.+4\Gamma_{-3,2}^{(2)})\right). && \text{(190h)}
\end{aligned}
$$

The results are identical to those obtained by method of reflections in Eqs. (119a), (119b), and (119c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

4. Y^C Function

The function Y^C gives the torque for the rotating particles with $m = 1$. Therefore, it is derived by the coefficient Q_{1pq} for the torque from the same recurrence relations for Y^A with a different initial condition

$$
P_{n00}^{(\alpha)} = 0, \quad V_{n00}^{(\alpha)} = 0, \quad Q_{n00}^{(\alpha)} = \delta_{1n}\Gamma_{0,3}^{(\alpha)}. \qquad \text{(191)}
$$

In this case, the coefficient $f_k^{YC\alpha}$ is defined by

$$
f_k^{YC\alpha} := 2^k \sum_{q=0}^{k} Q_{1(k-q)q}^{(\alpha)}\lambda^{q+j}, \qquad \text{(192)}
$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 7$ are

$$
\begin{aligned}
f_0^{YC1} &= \left(\Gamma_{0,3}^{(1)}\right), && \text{(193a)}\\
f_1^{YC1} &= 0, && \text{(193b)}\\
f_2^{YC1} &= 0, && \text{(193c)}\\
f_3^{YC1} &= \lambda^3\left(4\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)}\right), && \text{(193d)}\\
f_4^{YC1} &= \lambda\left(12(\Gamma_{0,3}^{(1)})^2\Gamma_{2,3}^{(2)}\right), && \text{(193e)}\\
f_5^{YC1} &= \lambda^4\left(18\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\right), && \text{(193f)}\\
f_6^{YC1} &= \lambda^2\left(27(\Gamma_{0,3}^{(1)})^2\Gamma_{2,3}^{(1)}(\Gamma_{2,3}^{(2)})^2\right)\\
&+ \lambda^3\left(16(\Gamma_{0,3}^{(1)})^2(15\Gamma_{2,5}^{(2)} + \Gamma_{0,3}^{(2)})\right), && \text{(193g)}\\
f_7^{YC1} &= \lambda^4\left(72(\Gamma_{0,3}^{(1)})^2\Gamma_{0,3}^{(2)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^5\left(81\Gamma_{0,3}^{(1)}\Gamma_{0,3}^{(2)}(\Gamma_{2,3}^{(1)})^2(\Gamma_{2,3}^{(2)})^2/2\right)\\
&+ \lambda^6\left(72\Gamma_{0,3}^{(1)}(\Gamma_{0,3}^{(2)})^2\Gamma_{2,3}^{(1)}\right). && \text{(193h)}
\end{aligned}
$$

The results are identical to those obtained by method of reflections in Eqs. (130a) and (130b) for the terms containing one or two Γ's. The results reduce to those by Jeffrey and Onishi[13] in the no-slip limit $\widehat{\gamma} = 0$ and those by Keh and Chen[16] in the case of $\widehat{\gamma}_1 = \widehat{\gamma}_2$.

5. Y^H Function

With the same recurrence relations and the initial condition for Y^C, that is, for the rotating particles, the function Y^H is obtained from the coefficient P_{2pq} for the stresslet instead of Q_{1pq} for the torque.

In this case, the coefficient $f_k^{YH\alpha}$ is defined by

$$
f_k^{YH\alpha} := -\left(\frac{3}{8}\right)2^k \sum_{q=0}^{k} P_{2(k-q)q}^{(\alpha)}\lambda^{q+j}, \qquad \text{(194)}
$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 7$ are

$$
\begin{aligned}
f_0^{YH1} &= 0, && \text{(195a)}\\
f_1^{YH1} &= 0, && \text{(195b)}\\
f_2^{YH1} &= 0, && \text{(195c)}\\
f_3^{YH1} &= \lambda^3\left(10\Gamma_{0,3}^{(2)}\Gamma_{2,5}^{(1)}\right), && \text{(195d)}\\
f_4^{YH1} &= 0, && \text{(195e)}\\
f_5^{YH1} &= 0, && \text{(195f)}\\
f_6^{YH1} &= \lambda\left(24\Gamma_{0,3}^{(1)}\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^3\left(-40\Gamma_{0,3}^{(1)}\Gamma_{2,5}^{(1)}(5\Gamma_{2,5}^{(2)} - 2\Gamma_{0,3}^{(2)})\right), && \text{(195g)}\\
f_7^{YH1} &= \lambda^4\left(36\Gamma_{0,3}^{(2)}\Gamma_{0,5}^{(1)}\Gamma_{2,3}^{(1)}\Gamma_{2,3}^{(2)}\right)\\
&+ \lambda^6\left(180(\Gamma_{0,3}^{(2)})^2\Gamma_{2,3}^{(1)}\Gamma_{2,5}^{(1)}\right). && \text{(195h)}
\end{aligned}
$$

The results are identical to those obtained by method of reflections in Eqs. (136a) and (136b) for the terms containing one or two Γ's. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

6. Y^M Function

The function Y^M gives the stresslet under a shear flow for $m = 1$. Therefore, it is derived by the coefficient P_{2pq} for the stresslet from the same recurrence relations for Y^A with a different initial condition. The boundary conditions are

$$\chi_{mn}^{(\alpha)} = \frac{2}{3}(-1)^{\alpha} a_\alpha E_\alpha \delta_{1m}\delta_{2n}, \quad (196a)$$

$$\psi_{mn}^{(\alpha)} = \frac{2}{3}(-1)^{\alpha} a_\alpha E_\alpha (1-6\widehat{\gamma})\delta_{1m}\delta_{2n}, \quad (196b)$$

$$\omega_{mn}^{(\alpha)} = 0, \quad (196c)$$

which correspond to Eq. (J-54) with the correction due to the slip. The expansions used here are [in (J-55), (J-56), and (J-57)]

$$p_{1n}^{(\alpha)} = (-1)^{\alpha}\frac{10}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq} t_\alpha^p t_{3-\alpha}^q, \quad (197a)$$

$$v_{1n}^{(\alpha)} = (-1)^{\alpha}\frac{10}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{V_{npq}}{2(2n+1)} t_\alpha^p t_{3-\alpha}^q, \quad (197b)$$

$$q_{1n}^{(\alpha)} = -\mathrm{i}\frac{10}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{npq} t_\alpha^p t_{3-\alpha}^q. \quad (197c)$$

The initial conditions are given from Eqs. (160a), (160b), and (160c) by

$$P_{n00}^{(\alpha)} = \delta_{n2}\Gamma_{2,5}^{(\alpha)}, \quad V_{n00}^{(\alpha)} = \delta_{n2}\Gamma_{0,5}^{(\alpha)}, \quad Q_{n00}^{(\alpha)} = 0. \quad (198)$$

In this case, the coefficient $f_k^{YM\alpha}$ is defined by

$$f_k^{YM\alpha} := 2^k \sum_{q=0}^{k} P_{2(k-q)q}^{(\alpha)} \lambda^{q+j}, \quad (199)$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 7$ are

$$f_0^{YM1} = \left(\Gamma_{2,5}^{(1)}\right), \quad (200a)$$

$$f_1^{YM1} = 0, \quad (200b)$$

$$f_2^{YM1} = 0, \quad (200c)$$

$$f_3^{YM1} = \lambda^3\left(-20\Gamma_{2,5}^{(1)}\Gamma_{2,5}^{(2)}\right), \quad (200d)$$

$$f_4^{YM1} = 0, \quad (200e)$$

$$f_5^{YM1} = \lambda^3\left(128\Gamma_{0,5}^{(1)}\Gamma_{2,5}^{(2)}\right) + \lambda^5\left(128\Gamma_{0,5}^{(2)}\Gamma_{2,5}^{(1)}\right), \quad (200f)$$

$$f_6^{YM1} = \lambda^3\left(80(\Gamma_{2,5}^{(1)})^2(5\Gamma_{2,5}^{(2)} + 3\Gamma_{0,3}^{(2)})\right), \quad (200g)$$

$$f_7^{YM1} = 0. \quad (200h)$$

The results are identical to those obtained by method of reflections in Eqs. (148a), (148b), and (148c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

E. Z Functions ($m = 2$)

The boundary conditions are given by

$$\chi_{mn}^{(\alpha)} = \frac{1}{3}(-1)^{3-\alpha} a_\alpha E_\alpha \delta_{2m}\delta_{2n}, \quad (201a)$$

$$\psi_{mn}^{(\alpha)} = \frac{1}{3}(-1)^{3-\alpha} a_\alpha E_\alpha (1-6\widehat{\gamma})\delta_{2m}\delta_{2n}, \quad (201b)$$

$$\omega_{mn}^{(\alpha)} = 0, \quad (201c)$$

which correspond to Eq. (J-69) with the correction due to the slip. The expansions used here are [in (J-70), (J-71), and (J-72)]

$$p_{2n}^{(\alpha)} = (-1)^{3-\alpha}\frac{5}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} P_{npq} t_\alpha^p t_{3-\alpha}^q, \quad (202a)$$

$$v_{2n}^{(\alpha)} = (-1)^{3-\alpha}\frac{5}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{V_{npq}}{2(2n+1)} t_\alpha^p t_{3-\alpha}^q, \quad (202b)$$

$$q_{2n}^{(\alpha)} = \mathrm{i}\frac{5}{3} a_\alpha E_\alpha \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} Q_{npq} t_\alpha^p t_{3-\alpha}^q. \quad (202c)$$

From Eqs. (161a), (161b), and (161c), substituting $m = 2$ and the above expansions, the recurrence relations are given by

$$\begin{aligned} P_{npq}^{(\alpha)} = \sum_{s=2}^{\infty}\binom{n+s}{n+2} &\times\left[-\frac{2(2n+1)(2n-1)}{n+1}\Gamma_{2,2n+1}^{(\alpha)} Q_{s(q-s-1)(p-n+1)}^{(3-\alpha)}\right. \\ &+\frac{n(2n+1)(2n-1)}{2(n+1)(2s+1)}\Gamma_{2,2n+1}^{(\alpha)} V_{s(q-s-2)(p-n+1)}^{(3-\alpha)} \\ &+\frac{2n+1}{n+1}\frac{ns(n+s-2ns-2)-2^2(2ns-4s-4n+2)}{2s(2s-1)(n+s)} \\ &\times\Gamma_{2,2n+1}^{(\alpha)} P_{s(q-s)(p-n+1)}^{(3-\alpha)} \\ &\left.+\frac{n(2n-1)}{2(n+1)}\Gamma_{0,2n+1}^{(\alpha)} P_{s(p-s)(p-n-1)}^{(3-\alpha)}\right], \end{aligned} \quad (203a)$$

$$V_{npq}^{(\alpha)} = \Gamma_{0,2}^{(\alpha)} P_{npq}^{(\alpha)} + \sum_{s=2}^{\infty}\binom{n+s}{n+2} \times\frac{2n}{(n+1)(2n+3)}\Gamma_{-(2n+1),2}^{(\alpha)} P_{s(q-s)(p-n-1)}^{(3-\alpha)}, \quad (203b)$$

$$\begin{aligned} Q_{npq}^{(\alpha)} = \sum_{s=2}^{\infty}\binom{n+s}{n+2} &\times\left[\frac{s}{(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} Q_{s(q-s-1)(p-n)}^{(3-\alpha)}\right. \\ &\left.-\frac{2}{ns(n+1)}\Gamma_{-(n-1),n+2}^{(\alpha)} P_{s(q-s)(p-n)}^{(3-\alpha)}\right]. \end{aligned} \quad (203c)$$

The initial conditions are obtained from Eqs. (160a), (160b), and (160c) as

$$P_{n00}^{(\alpha)} = \delta_{2n}\Gamma_{2,5}^{(\alpha)}, \quad V_{n00}^{(\alpha)} = \delta_{2n}\Gamma_{0,5}^{(\alpha)}, \quad Q_{n00}^{(\alpha)} = 0. \quad (204)$$

In this case, the coefficient $f_k^{ZM\alpha}$ is defined by

$$f_k^{ZM\alpha} := 2^k \sum_{q=0}^{k} P^{(\alpha)}_{2(k-q)q} \lambda^{q+j}, \tag{205}$$

where $j = 0$ for even k and $j = 1$ for odd k. The explicit forms up to $k = 11$ are

$$f_0^{ZM1} = \left(\Gamma^{(1)}_{2,5}\right), \tag{206a}$$
$$f_1^{ZM1} = 0, \tag{206b}$$
$$f_2^{ZM1} = 0, \tag{206c}$$
$$f_3^{ZM1} = 0, \tag{206d}$$
$$f_4^{ZM1} = 0, \tag{206e}$$
$$\begin{aligned} f_5^{ZM1} &= \lambda^3 \left(32\Gamma^{(1)}_{0,5}\Gamma^{(2)}_{2,5}\right) \\ &+ \lambda^5 \left(32\Gamma^{(2)}_{0,5}\Gamma^{(1)}_{2,5}\right), \end{aligned} \tag{206f}$$
$$f_6^{ZM1} = 0, \tag{206g}$$
$$f_7^{ZM1} = 0, \tag{206h}$$
$$f_8^{ZM1} = \lambda^5 \left(160(\Gamma^{(1)}_{2,5})^2(7\Gamma^{(2)}_{2,7} + 8\Gamma^{(2)}_{-1,4})/3\right), \tag{206i}$$
$$f_9^{ZM1} = 0, \tag{206j}$$
$$\begin{aligned} f_{10}^{ZM1} &= \lambda^3 \left(1024(\Gamma^{(1)}_{0,5})^2\Gamma^{(2)}_{2,5}\right) \\ &+ \lambda^5 \left(-256\Gamma^{(1)}_{0,5}\Gamma^{(1)}_{2,5}(35\Gamma^{(2)}_{2,7} - 8\Gamma^{(2)}_{0,5})\right) \\ &+ \lambda^7 \left(128(\Gamma^{(1)}_{2,5})^2(1620\Gamma^{(2)}_{2,9} - 525\Gamma^{(2)}_{0,2}\Gamma^{(2)}_{2,7} - 525\Gamma^{(2)}_{0,7} \right. \\ &\left. +168\Gamma^{(2)}_{0,2}\Gamma^{(2)}_{0,5} + 700\Gamma^{(2)}_{-2,5} + 32\Gamma^{(2)}_{-5,2})/21\right), \end{aligned} \tag{206k}$$
$$f_{11}^{ZM1} = 0. \tag{206l}$$

The results are identical to those obtained by method of reflections in Eqs. (153a), (153b), and (153c) for the terms containing one or two Γ's. The results reduce to those by Jeffrey[14] in the no-slip limit $\widehat{\gamma} = 0$.

V. CONCLUDING REMARKS

We have extended the calculations of resistance functions of two spheres with arbitrary size by the method of twin multipole expansions in linear flows by Jeffrey and Onishi[13] and Jeffrey[14] to the slip particles with Navier's slip boundary condition with arbitrary slip lengths. This complements the previous results of slip particles by Keh and Chen[16] for the same scaled slip lengths without strain flows. The present results have confirmed to recover these existing results, that is, those by Jeffrey *et al.*[13,14] in the no-slip limit, and those by Keh and Chen[16] in the case of equal scaled slip lengths. We have also derived the resistance functions by method of reflections and confirmed the consistency with the twin multipole expansions.

The present solutions of two-sphere problem cover much wider range than the previous solutions. Because the particle radii and slip lengths can be chosen independently, the solutions cover not only the problem of two bubbles (demonstrated in Keh and Chen[16]) but also pair of solid particle and gas bubble, for example, with arbitrary sizes.

In addition to these fundamental aspects in fluid mechanics, the solutions of slip particles is quite important for applications such as micro- and nanofluidics, where the no-slip boundary condition may break. This should be stressed, especially from the fact that the solutions of slip boundary condition are relatively limited to the no-slip case.

Using the multipole expansions and Faxén's laws derived in the present paper, recently the Stokesian dynamics method[22] is extended to the slip particles.[12] Because the lubrication corrections are missing in the formulation, the applicability is limited to relatively dilute configurations. The present work is a first step of an attempt to improve the Stokesian dynamics method for slip particles at the level of the no-slip particles. Unfortunately, the attempt is not achieved yet, because of the lack of lubrication theory for resistance functions of slip particles (with some exception[10]). Although the present solutions are in the form of $1/r$ expansion, they are complete set, that is, all 11 scalar functions for all pairs of particles $\alpha\beta$. It is hoped that the present study would help completing lubrication theory and the Stokesian dynamics method for arbitrary slip particles.

The computer programs used in the paper and the results of coefficients in mathematical form for higher orders (up to $k = 20$) will be available on the open source project "RYUON-twobody" (`http://ryuon.sourceforge.net/twobody/`).

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