Closure relations for non-uniform suspensions

Kengo Ichiki
Andrea Prosperetti

Department of Mechanical Engineering
The Johns Hopkins University

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Introduction

Non-uniform suspensions .

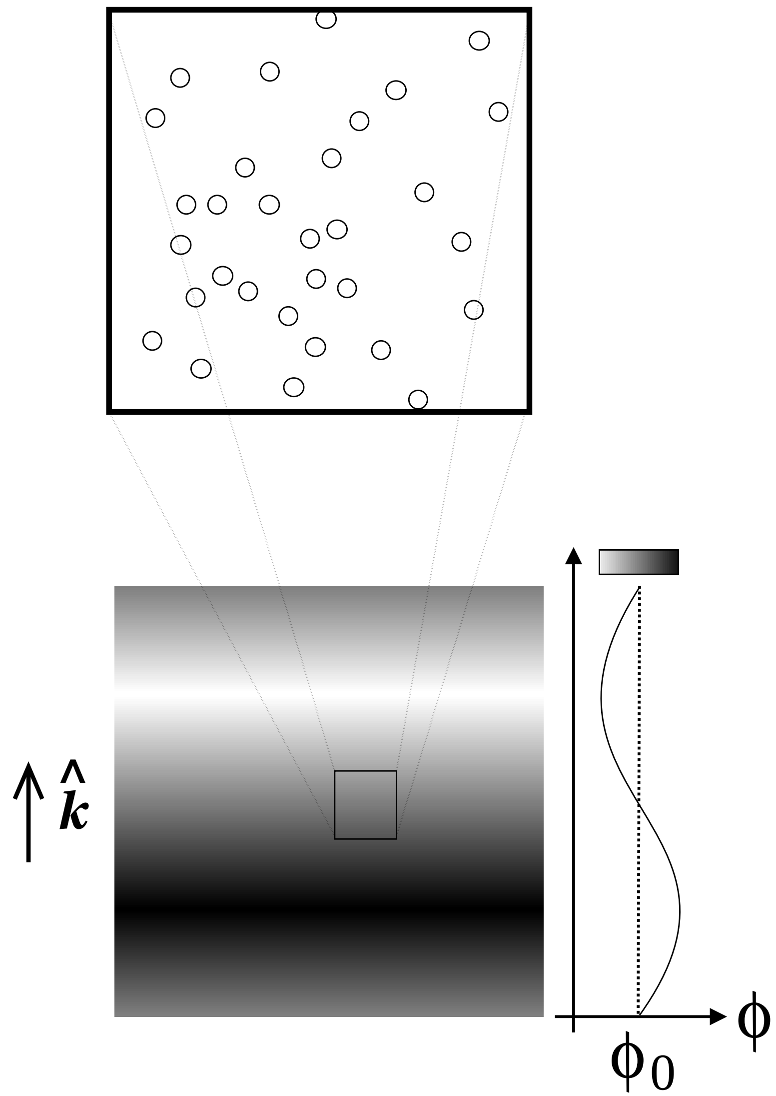
Practically important:

- Shear-induced diffusivity
- Particle migration in Stokes flows
- Stratification in sedimentation

Uniform suspension is too simple:

- No strain in sedimentation
- No slip velocity in shear problem

Important physics vanishes!



Introduction

Goal: To derive the **constitutive equations** of

S : viscous stress of the mixture

$\boldsymbol{F}$: interphase force

valid for all **sedimentation**, **torque**, and **shear** problems
from first-principle simulations

by Stokesian Dynamics method (Mo-Sangani 1994)

under periodic boundary condition

for random hard-sphere configurations with **non-uniform** weight

References:

Marchioro *et al.*, *Int. J. Multiphase Flow* **26** (2000) 783; **27** (2001) 237.

Ichiki and Prosperetti, submitted to *Phys. Fluids*.

Rheology

Uniform suspensions — Shear

$$\frac{\mathsf{S}}{\mu} = 2\,\mu_e\,\mathsf{E}_m$$

$$\mathsf{E}_m = \frac{1}{2}\left[\nabla \boldsymbol{u}_m + (\nabla \boldsymbol{u}_m)^\dagger\right]$$

$\boldsymbol{u}_m$: mixture velocity

μ : viscosity of the fluid

μ_e : relative viscosity of the mixture

Non-uniform suspensions – Sedimentation

$\mathsf{E}_m \neq 0$ and μ_e plays a role

Viscous Stress S

Closure relation

$$\frac{\mathsf{S}}{\mu} = 2\,\mu_e\,\mathsf{E}_m + 2\,\mu_\Delta\,\mathsf{E}_\Delta + 2\,\mu_\nabla\,\mathsf{E}_\nabla$$

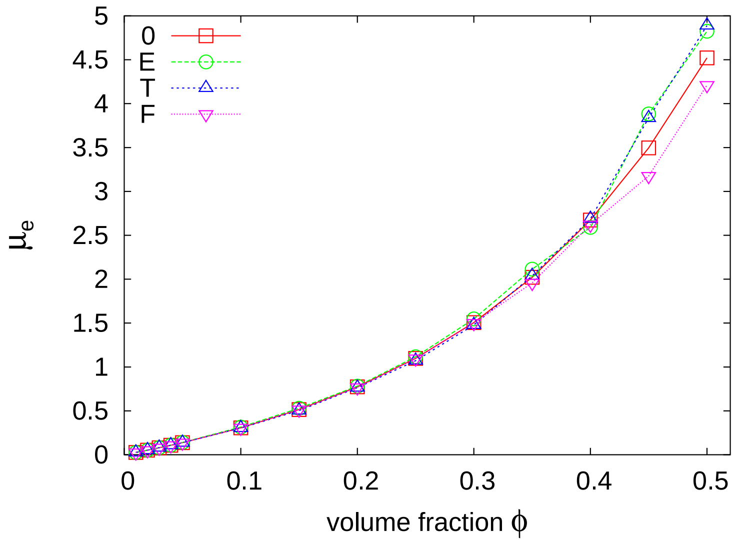
$$\mathsf{E}_\Delta = \frac{1}{2}\left[\nabla \boldsymbol{u}_\Delta + (\nabla \boldsymbol{u}_\Delta)^\dagger\right] - \frac{1}{3}(\nabla\cdot\boldsymbol{u}_\Delta)\,\mathsf{I}$$

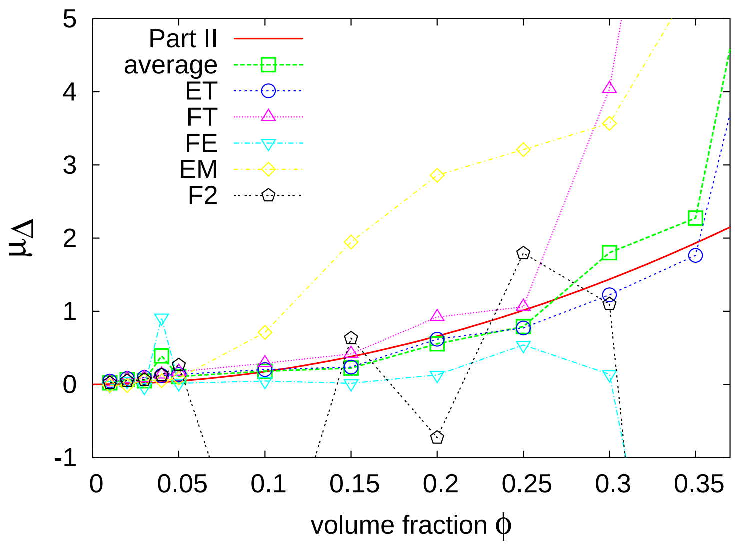
$$\mathsf{E}_\nabla = \frac{1}{2}\left[\boldsymbol{u}_\Delta\nabla\phi + (\boldsymbol{u}_\Delta\nabla\phi)^\dagger\right] - \frac{1}{3}(\boldsymbol{u}_\Delta\cdot\nabla\phi)\,\mathsf{I}$$

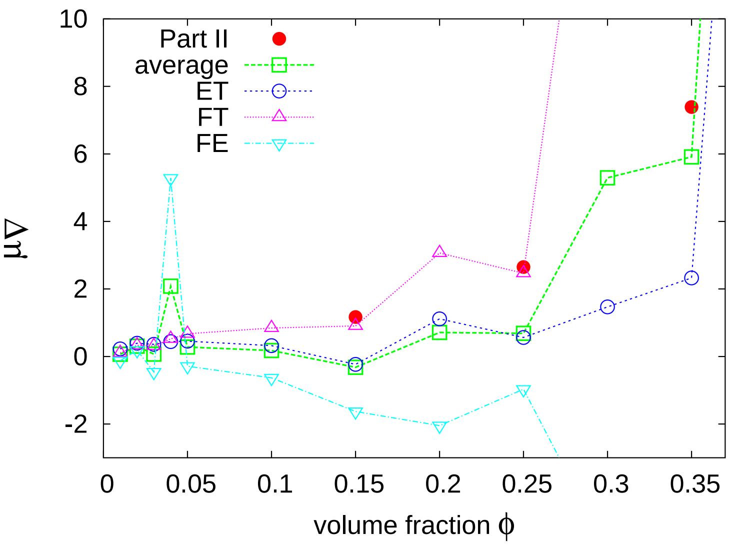
$\boldsymbol{u}_\Delta$: slip velocity

ϕ : volume fraction

Viscous Stress S – Results







Sedimentation

Uniform suspensions — Sedimentation

$$\boldsymbol{u}_\Delta = U(\phi)\,\frac{\boldsymbol{F}}{6\pi\mu a}$$

$\boldsymbol{u}_\Delta$: slip velocity

$U(\phi)$: hindrance function

$\boldsymbol{F}$: interphase force

Non-uniform suspensions — Shear

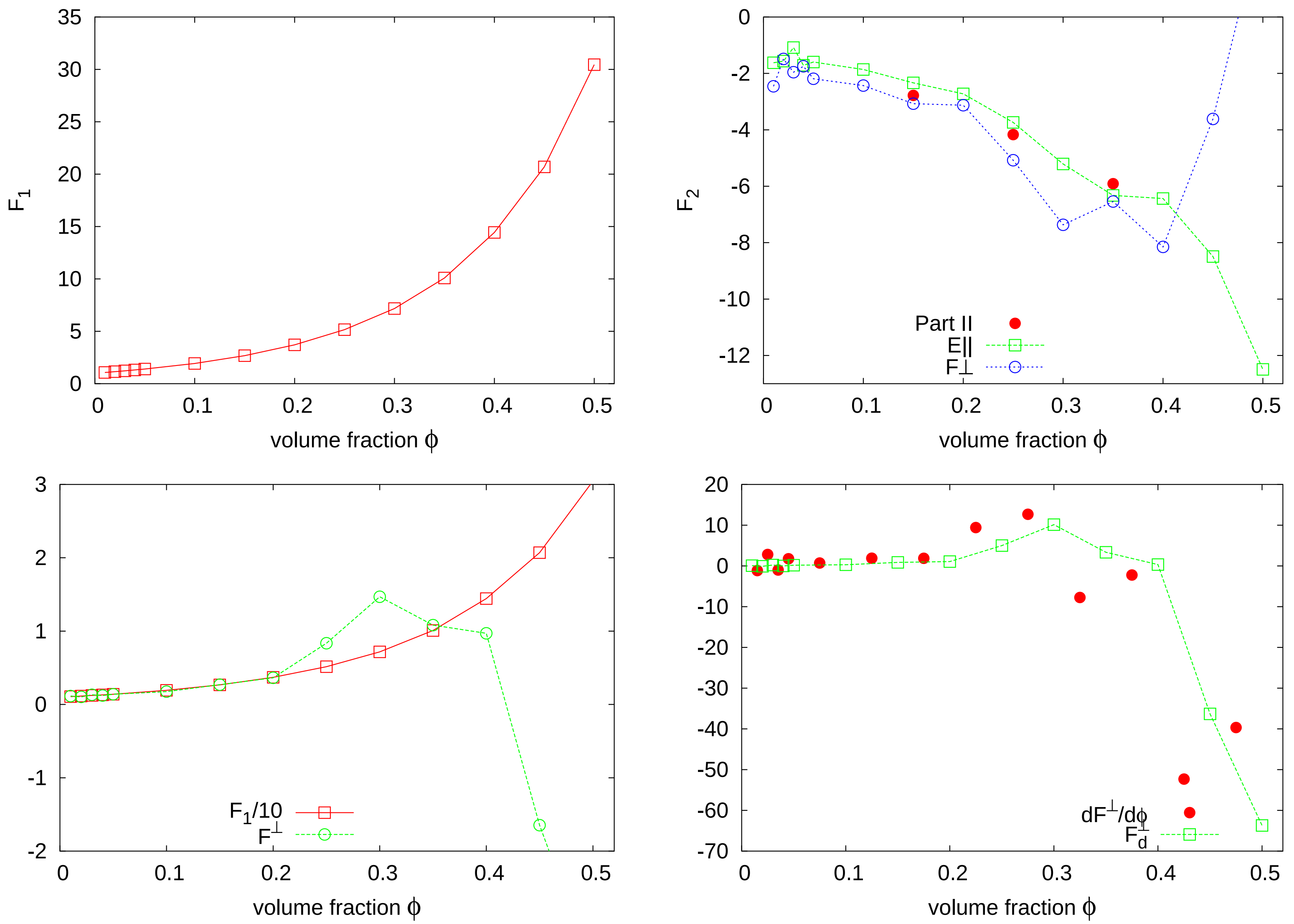
$$\boldsymbol{F} = 0 \quad \text{but} \quad \boldsymbol{u}_\Delta \neq 0$$

Interphase Force $\boldsymbol{F}$

Closure relation

$$
\begin{aligned}
\frac{\boldsymbol{F}}{6\pi\mu a} &= F_1 \;\boldsymbol{u}_\Delta \\
&+ F_2 \; a^2\, \mathsf{E}_m \cdot \boldsymbol{\nabla}\phi \\
&+ F_3 \; a^2 \nabla^2 \boldsymbol{u}_m \\
&+ F_4 \; a^2\, \boldsymbol{\nabla} \times \boldsymbol{\Omega}_\Delta \\
&+ F_5 \; a^2 \,(\boldsymbol{\nabla}\phi) \times \boldsymbol{\Omega}_\Delta \\
&+ F^{\perp} \; a^2 \left(\nabla^2 \mathsf{I} - \boldsymbol{\nabla}\boldsymbol{\nabla}\right) \cdot \boldsymbol{u}_\Delta \\
&+ F_d^{\perp} \; a^2\, \boldsymbol{u}_\Delta \cdot \left(\nabla^2 \mathsf{I} - \boldsymbol{\nabla}\boldsymbol{\nabla}\right)\phi \\
&+ F^{\parallel} \; a^2\, \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}_\Delta \\
&+ F_d^{\parallel} \; a^2\, \boldsymbol{u}_\Delta \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla}\phi)
\end{aligned}
$$

Interphase Force F – Results



Interphase Force $\boldsymbol{F}$

Closure relation

Results and suggestions

$$\frac{\boldsymbol{F}}{6\pi\mu a} = F_1 \; \boldsymbol{u}_\Delta \qquad F_1 = 1/U(\phi)$$

$$+ F_2 \; a^2 \, \mathsf{E}_m \cdot \boldsymbol{\nabla}\phi \qquad F_2 \approx 2 \, \mathrm{d}F_3/\mathrm{d}\phi$$

$$+ F_3 \; a^2 \nabla^2 \boldsymbol{u}_m$$

$$+ F_4 \; a^2 \, \boldsymbol{\nabla} \times \boldsymbol{\Omega}_\Delta$$

$$+ F_5 \; a^2 \, (\boldsymbol{\nabla}\phi) \times \boldsymbol{\Omega}_\Delta \qquad F_5 \approx \mathrm{d}F_4/\mathrm{d}\phi$$

$$+ F^\perp \; a^2 \left(\nabla^2 \mathsf{I} - \boldsymbol{\nabla}\boldsymbol{\nabla}\right) \cdot \boldsymbol{u}_\Delta \qquad F^\perp \approx F_1/10 \text{ for small } \phi$$

$$+ F_d^\perp \; a^2 \, \boldsymbol{u}_\Delta \cdot \left(\nabla^2 \mathsf{I} - \boldsymbol{\nabla}\boldsymbol{\nabla}\right) \phi \qquad F_d^\perp \approx \mathrm{d}F^\perp/\mathrm{d}\phi$$

$$+ F^\parallel \; a^2 \, \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}_\Delta \qquad F^\parallel = F_1/10$$

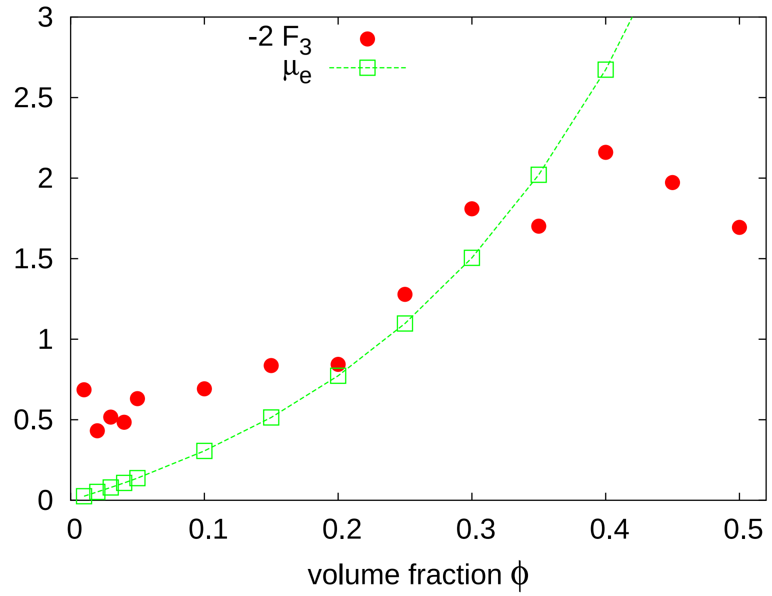
$$+ F_d^\parallel \; a^2 \, \boldsymbol{u}_\Delta \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla}\phi) \qquad F_d^\parallel = 0$$

Discussions

Expected constitutive equation of $\boldsymbol{F}$:

$$\frac{\boldsymbol{F}}{6\pi\mu a} = \left(1 + \frac{a^2\nabla^2}{10}\right)(F_1\, \boldsymbol{u}_\Delta) + a^2 \boldsymbol{\nabla} \cdot (2\, F_3\, \mathsf{E}_m) + a^2 \boldsymbol{\nabla} \times (F_4\, \boldsymbol{\Omega}_\Delta)$$

This suggests a relation between μ_e and F_3:



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Conclusions

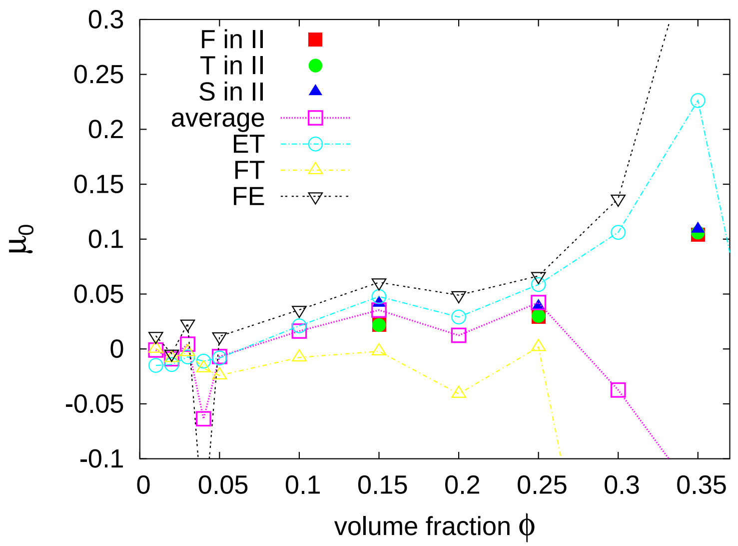
- develop a systematic closure procedure for non-uniform suspensions
- apply it to S and $\boldsymbol{F}$
- derive the constitutive equations, determine all closure coefficients systematically, valid for both uniform and non-uniform suspensions and for all sedimentation, torque, and shear problems

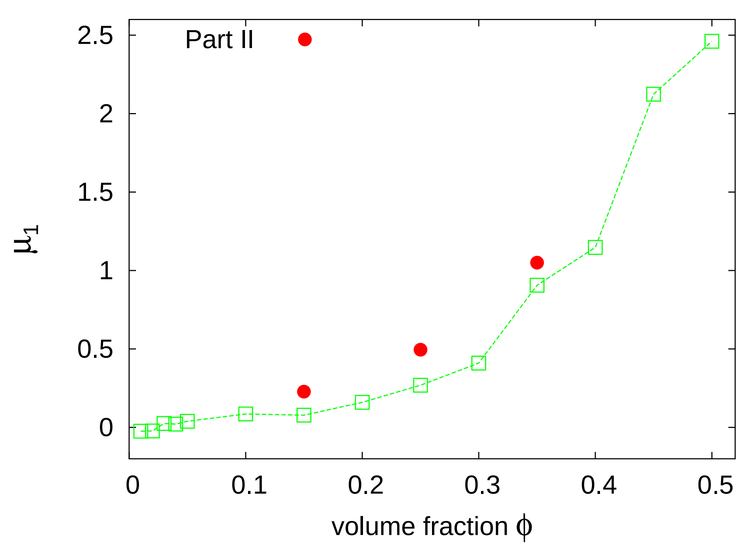
Future plans:

- apply the closure procedure to interphase torque $\boldsymbol{T}$ and anti-symmetric part of the stress $\boldsymbol{V}$
- study the relation among the closure coefficients

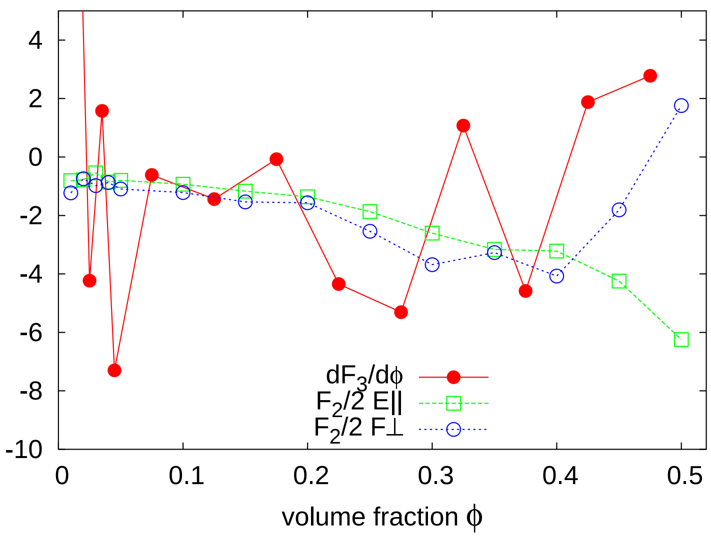
More Results of S

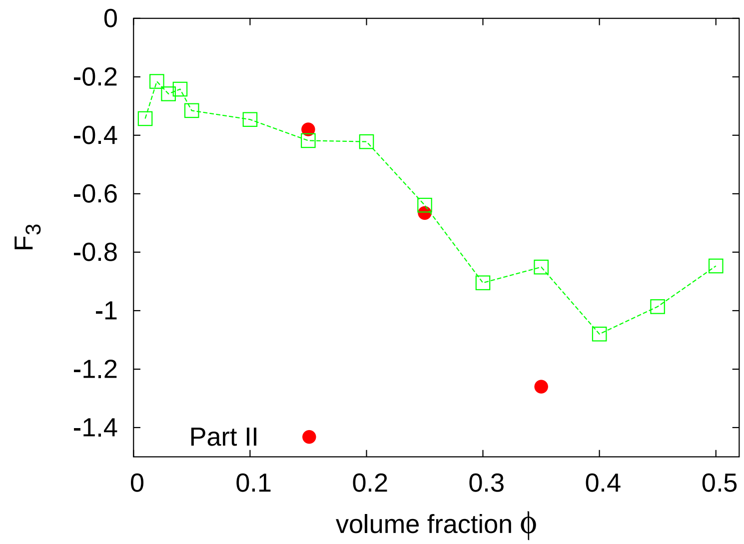
$$\frac{S}{\mu} = 2\,\mu_e\,E_m + 2\,\mu_\Delta\,E_\Delta + 2\,\mu_\nabla\,E_\nabla + 2\,\mu_0\,a^2\nabla^2 E_\nabla + 2\,\mu_1\,a^2 E_\nabla\left(\nabla^2\phi\right)$$

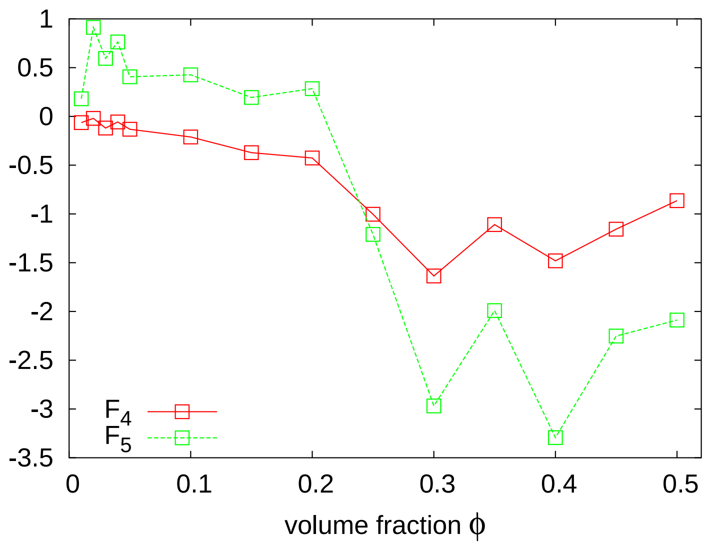


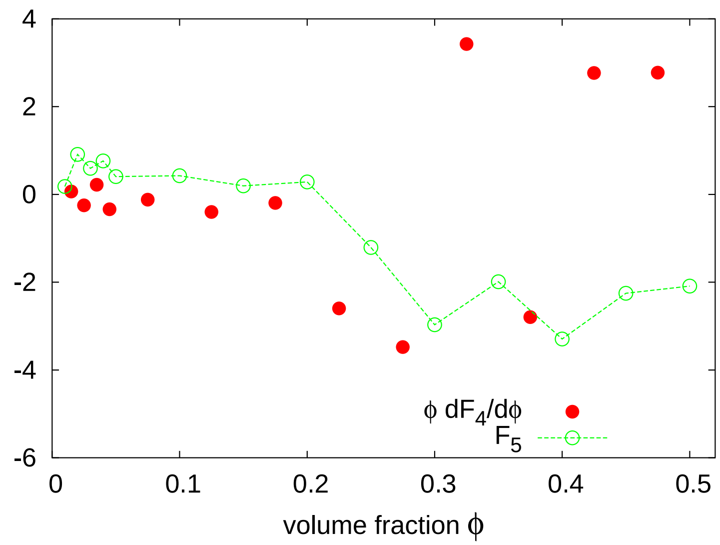


More Results of F

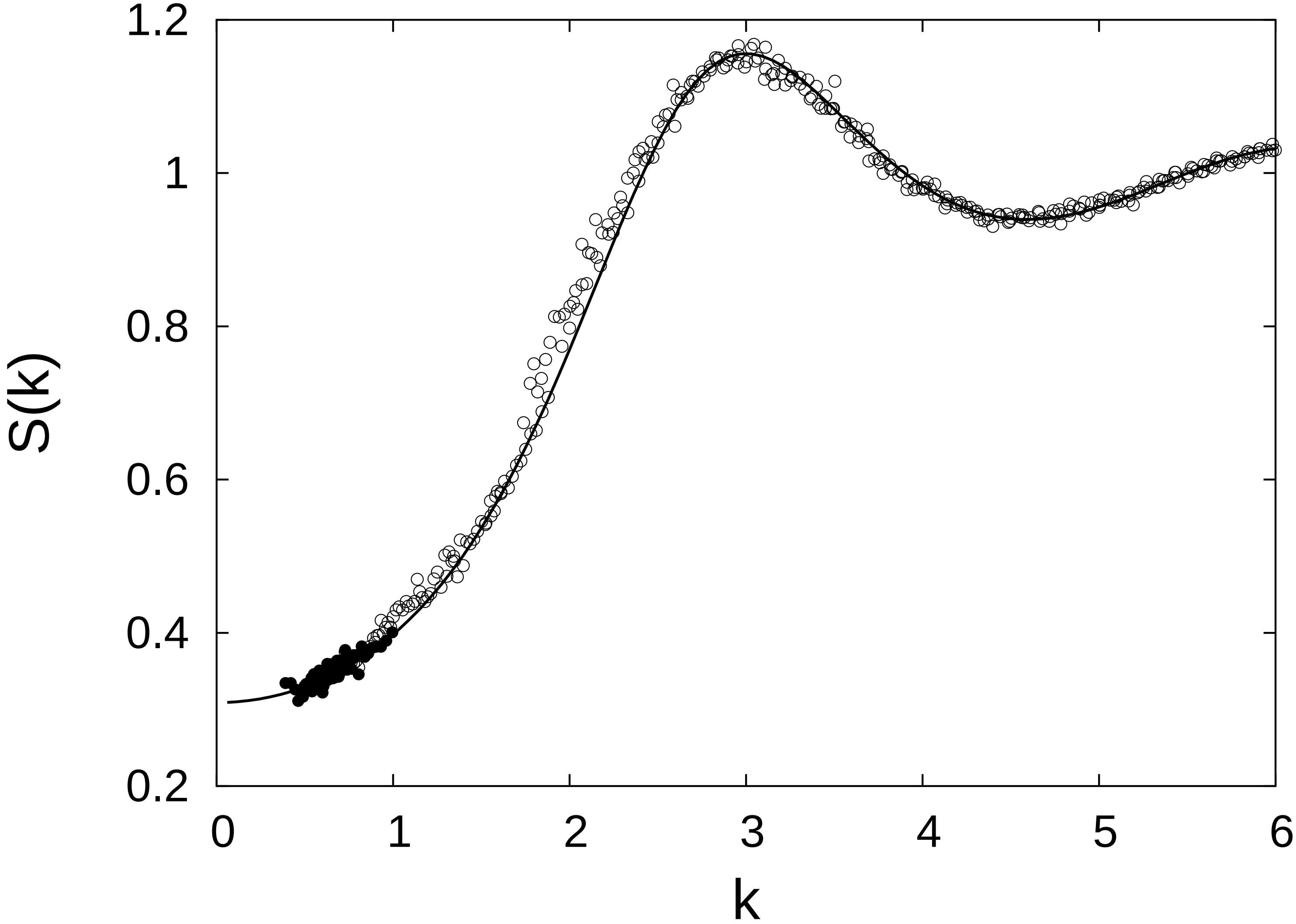




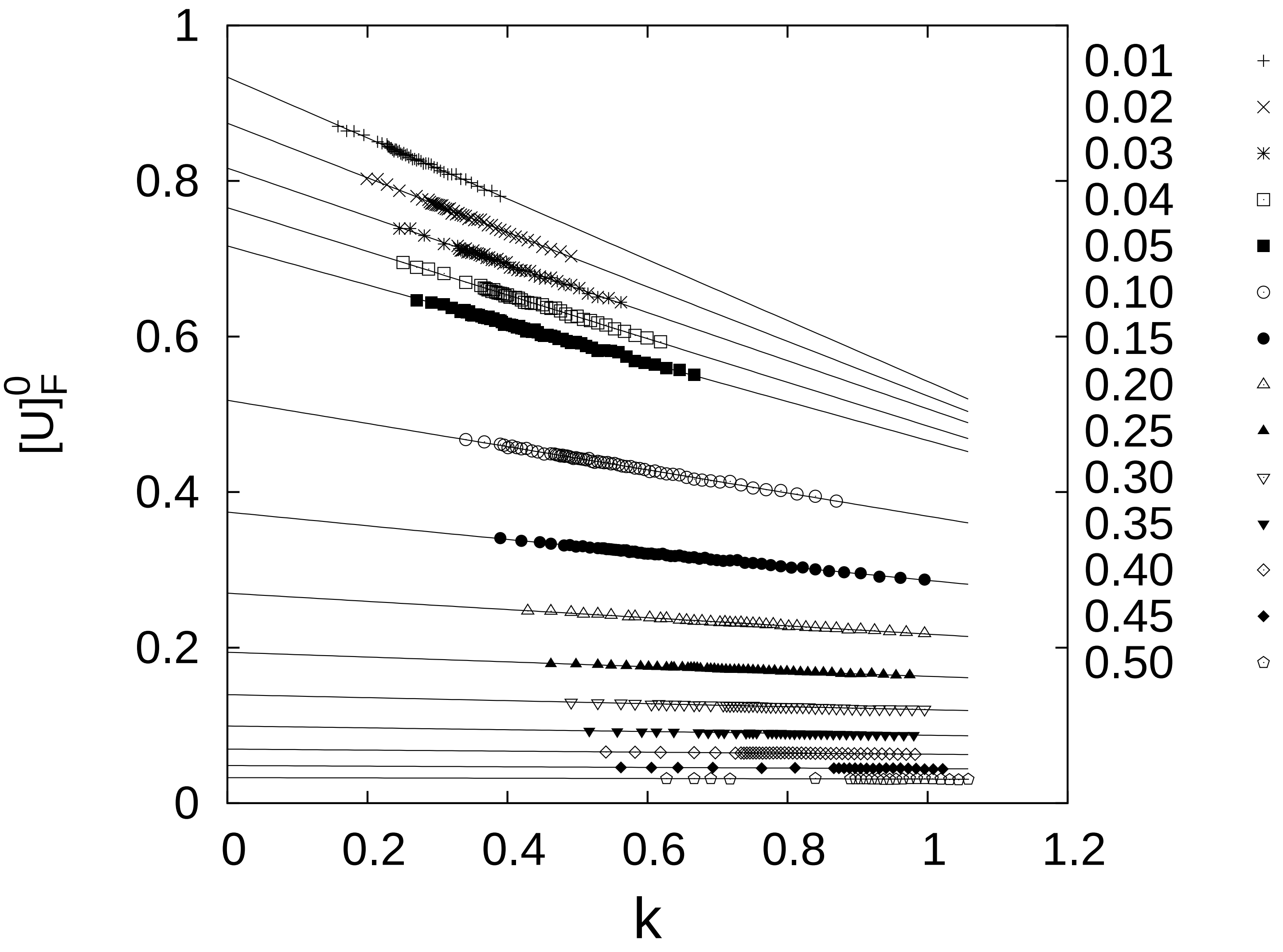




Structure Factor $S(k)$



Averages and Fitting



Closure equations of F

$$[F]_F^0 = F_1 [u_\Delta]_F^0$$

$$[F]_F^{\parallel} = \left(F_1 - k^2 F^{\parallel}\right)[u_\Delta]_F^{\parallel} + \phi\left(1 - \frac{k^2}{10}\right)\left(\frac{\mathrm{d}F_1}{\mathrm{d}\phi} - k^2 F_d^{\parallel}\right)[u_\Delta]_F^0$$

$$[F]_F^{\perp} = \left(F_1 - k^2 F^{\perp}\right)[u_\Delta]_F^{\perp} + \phi\left(1 - \frac{k^2}{10}\right)\left(\frac{\mathrm{d}F_1}{\mathrm{d}\phi} - k^2 F_d^{\perp}\right)[u_\Delta]_F^0$$
$$-F_3 k^2 [u_m]_F^{\perp} + F_4 k [\Omega_\Delta]_F^{\perp}$$

$$[F]_T^{\perp} = \left(F_1 - k^2 F^{\perp}\right)[u_\Delta]_T^{\perp} - F_3 k^2 [u_m]_T^{\perp} + F_4 k [\Omega_\Delta]_T^{\perp}$$
$$+F_5 \phi\left(1 - \frac{k^2}{10}\right) k [\Omega_\Delta]_T^0$$

$$[F]_E^{\parallel} = \left(F_1 - k^2 F^{\parallel}\right)[u_\Delta]_E^{\parallel} + F_2 k \phi\left(1 - \frac{k^2}{10}\right)$$

$$[F]_E^{\perp} = \left(F_1 - k^2 F^{\perp}\right)[u_\Delta]_E^{\perp} - F_3 k^2 [u_m]_E^{\perp} + F_2 k \phi\left(1 - \frac{k^2}{10}\right) - F_4 k [\Omega_\Delta]_E^{\perp}$$